

CLASP TECHNOLOGY TO KNOT HOMOLOGY VIA THE AFFINE GRASSMANNIAN

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ABSTRACT. We use the affine Grassmannian to categorify the Reshetikhin-Turaev tangle invariants of type A. Our main tool is a categorification of all the generalized Jones-Wenzl projectors (a.k.a. clasps) as infinite twists. Applying this to certain convolution product varieties on the affine Grassmannian we extend our earlier work with Kamnitzer [CaK1, CaK2] from standard to arbitrary representations.

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1. INTRODUCTION

The purpose of this paper is twofold. First we give a general method based on the higher representation theory of $U_q(\mathfrak{sl}_\infty)$ for categorifying all the tangle invariants associated to \mathfrak{sl}_m . Roughly speaking, we show how to define homological tangle invariants starting with any categorification of the special $U_q(\mathfrak{sl}_\infty)$ -modules $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ (or of its submodule $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$). Along the way we categorify the Jones-Wenzl projectors and their generalizations (called clasps).

Secondly, we explain how the affine Grassmannian of PGL_m (resp. Nakajima quiver varieties) gives rise to a categorification of the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ (resp. $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$) by using categories of coherent sheaves. Subsequently, both the affine Grassmannian and Nakajima quiver varieties can be used to categorify all the Reshetikhin-Turaev link invariants of type A.

1.1. Reshetikhin-Turaev invariants and Jones-Wenzl projectors. Let \mathfrak{g} be a complex semisimple Lie algebra and denote by $U_q(\mathfrak{g})$ the corresponding quantum group. Consider a tangle T whose strands are labeled by dominant weights of \mathfrak{g} so that the strands at the bottom and top are labeled by $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\underline{\mu} = (\mu_1, \dots, \mu_{n'})$ respectively.

Following Reshetikhin-Turaev [RT] one can associate to T a map of $U_q(\mathfrak{g})$ -modules

$$\psi(T) : V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \longrightarrow V_{\mu_1} \otimes \cdots \otimes V_{\mu_{n'}} = V_{\underline{\mu}}$$

where V_{λ} denotes the irreducible $U_q(\mathfrak{g})$ -module with highest weight λ . This map is an invariant of framed tangles. In particular, if $T = K$ is a framed link then $\psi(K)$ is an endomorphism of the trivial $U_q(\mathfrak{g})$ -module. Such a map is given as multiplication by some $\psi(K)(1) \in \mathbb{C}(q)$. For example, if $\mathfrak{g} = \mathfrak{sl}_2$ and all the strands are labeled by the standard representation then $\psi(K)(1)$ is just the Jones polynomial of K .

If $\mathfrak{g} = \mathfrak{sl}_m$ then to define $\psi(T)$ for a general tangle T we need the following:

- (i) maps $\psi(T)$ where the strands of T are labeled only by fundamental weights and
- (ii) idempotents corresponding to the composition $P\mathbf{1}_{\underline{i}}$

$$V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_N}} \xrightarrow{\pi} V_{\sum_k \Lambda_{i_k}} \xrightarrow{\iota} V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_N}}$$

where $\Lambda_1, \dots, \Lambda_{m-1}$ denote the fundamental weights of \mathfrak{sl}_m and π, ι are the natural projection and inclusion maps.

For example, to compute the invariant of the unknot labeled by $V_{2\Lambda_1}$ one calculates the following composition

$$\mathbb{C} \rightarrow (V_{\Lambda_1}^{\vee} \otimes V_{\Lambda_1}^{\vee}) \otimes (V_{\Lambda_1} \otimes V_{\Lambda_1}) \xrightarrow{(II)(P)} (V_{\Lambda_1}^{\vee} \otimes V_{\Lambda_1}^{\vee}) \otimes (V_{\Lambda_1} \otimes V_{\Lambda_1}) \rightarrow \mathbb{C}$$

where the leftmost (resp. rightmost) map corresponds to a double cup (resp. double cap).

One way to define the maps in (i) is using skew Howe duality (see section 6.1 for details). More precisely, fix m and consider the vector space $\Lambda^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$ equipped with commuting actions of $U(\mathfrak{sl}_m)$ and $U(\mathfrak{sl}_N)$. As a $U(\mathfrak{sl}_N)$ -module it breaks up into weight spaces of the form $V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_N}}$ where $E_k, F_k \in U(\mathfrak{sl}_{\infty})$ are maps

$$(1) \quad V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_k}} \otimes V_{\Lambda_{i_{k+1}}} \otimes \cdots \otimes V_{\Lambda_{i_N}} \xrightleftharpoons[F_k]{E_k} V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_{k-1}}} \otimes V_{\Lambda_{i_{k+1}+1}} \otimes \cdots \otimes V_{\Lambda_{i_N}}.$$

The generators s_k of the Weyl group of \mathfrak{sl}_N induce maps

$$(2) \quad s_k : V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_k}} \otimes V_{\Lambda_{i_{k+1}}} \otimes \cdots \otimes V_{\Lambda_{i_N}} \longrightarrow V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_{k+1}}} \otimes V_{\Lambda_{i_k}} \otimes \cdots \otimes V_{\Lambda_{i_N}}.$$

One can lift these maps from $U(\mathfrak{sl}_N)$ to $U_q(\mathfrak{sl}_N)$ to obtain a braid group action between the weight spaces of $\Lambda_q^N(\mathbb{C}^m \otimes \mathbb{C}^{2N})$. In [CKL1] we showed that this agrees with the braid group action defined by Reshetikhin-Turaev using the R-matrix associated with \mathfrak{sl}_m . With a little more work (see section 7.2) one can also define caps and cups and thus recover all the tangle maps $\psi(T)$ from (i).

Notice that more strands in our tangle the larger the N we need to use. In order to work with all tangles in a uniform manner we let $N \rightarrow \infty$ and pass to the $U_q(\mathfrak{sl}_{\infty})$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$.

Next consider the idempotents $P\mathbf{1}_{\underline{i}}$ in (ii). When $m = 2$ they are the standard *Jones-Wenzl projectors*. These have been studied extensively (see for instance [KaLi]) and can be described recursively in terms of caps and cups. When $m = 3$ they were studied by Kuperberg [Kup] and also given a (more complicated) recursive definition in terms of webs. Kuperberg called these idempotents *clasps*. We will adopt this terminology for all idempotents $P\mathbf{1}_i$ when $m > 2$ (alternatively one could call these *generalized Jones-Wenzl projectors*).

1.2. Categorification. The idea of categorification is to replace the vector spaces $V_{\underline{\lambda}}$ by additive categories $\mathcal{D}(\underline{\lambda})$ and the maps $\psi(T) : V_{\underline{\lambda}} \rightarrow V_{\underline{\mu}}$ by functors $\Psi(T) : \mathcal{D}(\underline{\lambda}) \rightarrow \mathcal{D}(\underline{\mu})$ such that:

- $\Psi(T)$ are tangle invariants (*i.e.* invariant under the Reidemeister moves),
- the Grothendieck group of $\mathcal{D}(\underline{\lambda})$ is isomorphic to $V_{\underline{\lambda}}$,
- $\Psi(T)$ induces the Reshetikhin-Turaev map $\psi(T)$ at the level of Grothendieck groups.

If $T = K$ is a link and $\mathcal{D}(0)$ (the category associated to the trivial representation) happens to be the homotopy category of graded vector spaces then $\Psi(K)$ is given by tensoring with a complex $\Psi(K)(\mathbb{C})$ of graded vector spaces. Thus $\Psi(K)(\mathbb{C})$ yields a \mathbb{Z}^2 -graded invariant of the link K whose graded Euler characteristic is the Reshetikhin-Turaev invariant $\psi(K)(1)$.

The first example of this type of categorification is due to Khovanov [Kh1, Kh2]. He considers the case where $\mathfrak{g} = \mathfrak{sl}_2$ and all the strands of T are labeled by the standard representation. His construction uses the derived category of modules over certain arc-algebras¹. In subsequent work [KR], Khovanov and Rozansky consider $\mathfrak{g} = \mathfrak{sl}_m$ where all strands are again labeled by the standard representation (or its dual). This time $\mathcal{D}(\underline{\lambda})$ are categories of matrix factorizations.

In [CaK1, CaK2], jointly with Kamnitzer, we consider $\mathfrak{g} = \mathfrak{sl}_m$ where again all strands are labeled by the standard representation but instead of matrix factorizations we use derived categories of coherent sheaves on certain varieties (see section 8.3.1). These varieties are iterated projective bundles and their construction was inspired by the geometric Satake correspondence for the affine Grassmannian of PGL_m .

The current paper extends [CaK1, CaK2] to tangles labeled by arbitrary representations of \mathfrak{sl}_m . To do this we lift the maps in (i) and (ii) from vector spaces to categories. To lift (i) we start with any categorical 2-representation \mathcal{K} of \mathfrak{sl}_∞ (see section 2.2) which categorifies the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$. Roughly, this means that we have additive categories $\mathcal{D}(\underline{i})$ categorifying the weight spaces $V_{\Lambda_{i_1}} \otimes \cdots \otimes V_{\Lambda_{i_N}}$ and functors

$$\mathcal{D}(i_1, \dots, i_k, i_{k+1}, \dots, i_N) \begin{array}{c} \xrightarrow{E_k} \\ \xleftarrow{F_k} \end{array} \mathcal{D}(i_1, \dots, i_k - 1, i_{k+1} + 1, \dots, i_N)$$

lifting the maps in (1). Using a categorical version of skew Howe duality (following [CKL1, CKL2, CKL3]) one can show that such a 2-representation gives rise to a braid group action

$$(3) \quad T_k : \text{Kom}(\mathcal{D}(i_1, \dots, i_k, i_{k+1}, \dots, i_N)) \longrightarrow \text{Kom}(\mathcal{D}(i_1, \dots, i_{k+1}, i_k, \dots, i_N))$$

which lifts the map in (2) from vector spaces to categories (here Kom denotes the associated homotopy categories as in Example 2 of section 3.1.3).

In step (ii) we need to categorify the clasps $P\mathbf{1}_{\underline{i}}$. The clasps for $\mathfrak{g} = \mathfrak{sl}_m$ when $m > 3$ have not been studied extensively and even when $m = 2, 3$ they are quite complicated. If $m = 2$ (the case of Jones-Wenzl projectors) there are several categorifications which we discuss in section 1.5.

To categorify more general clasps ($m > 2$) we use an observation which (to the best of our knowledge) first appeared in [Roz]. It relates the Jones-Wenzl projectors with infinite braids.

One of the main results of this paper is to show that this is true for arbitrary clasps. More precisely, consider the infinite braid $\lim_{\ell \rightarrow \infty} T_\omega^{2\ell} \mathbf{1}_{\underline{i}}$ where T_ω (see equation (6)) denotes the braid element corresponding to the longest root. We show that:

- [Theorem 2.2] This limit exists (*i.e.* that it converges) and defines an idempotent $P^{-}\mathbf{1}_{\underline{i}}$.
- [Theorem 2.3] $P^{-}\mathbf{1}_{\underline{i}}$ categorifies the clasp $P\mathbf{1}_{\underline{i}}$.
- [Theorem 2.4] Together with the tangle functors $\Psi(T)$ these categorical clasps define an invariant $\Psi_{-}(T)$ which yields a \mathbb{Z}^2 -graded link invariant $\mathcal{H}_{-}^{i,j}(K)$.

We can summarize the content of these three theorems as follows:

Any 2-representation of \mathfrak{sl}_∞ which categorifies $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ can be used to categorify the Reshetikhin-Turaev invariants of framed tangles labeled by arbitrary representations of \mathfrak{sl}_m .

¹Technically speaking the Grothendieck groups of these module categories are only isomorphic to the zero weight space of $V_{\underline{\lambda}}$ but for the purpose of categorifying link invariants this suffices.

1.3. The affine Grassmannian and Nakajima quiver varieties. To apply the constructions above we need to categorify the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$. Such a categorification is given using derived categories of coherent sheaves on certain varieties $Y(\underline{i})$ (see 8.3.1).

More precisely, we define the 2-category $\mathcal{K}_{\text{Gr},m}$ where the objects are $\mathcal{D}(\underline{i}) := D(Y(\underline{i}))$, 1-morphisms are integral kernels (Fourier-Mukai transforms) between these varieties and 2-morphisms are maps between such kernels. Then we show that:

- [Theorem 2.6] $\mathcal{K}_{\text{Gr},m}$ is a categorical 2-representation of \mathfrak{sl}_∞ in the sense of section 2.2 which categorifies the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$.

The $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ contains a natural submodule $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$ consisting of the weight zero weights for the commuting action of $U_q(\mathfrak{sl}_m)$. It turns out that:

- [Proposition 9.1] Derived categories of coherent sheaves on Nakajima quiver varieties of type A can be used to define a 2-category $\mathcal{K}_{\text{Q},m}$ which categorifies $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$.
- [Theorem 9.3] Any categorification of $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$ can be used, in the same way as above, to categorify the Reshetikin-Turaev link invariants of type A.

We can summarize all this as follows:

Starting with the affine Grassmannian of type A (resp. Nakajima quiver varieties of type A) we define a 2-category $\mathcal{K}_{\text{Gr},m}$ (resp. $\mathcal{K}_{\text{Q},m}$) which categorifies the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ (resp. $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$).

1.4. Vertex operator constructions. The Frenkel-Kac-Segal vertex operator construction recovers the basic representation of $\widehat{\mathfrak{sl}}_n$ from the Fock space representation of the Heisenberg algebra. This construction was categorified in [CLi1, CLi2].

More precisely, in [CLi1] we defined a 2-representation on categories of coherent sheaves on Hilbert schemes of points $\bigsqcup_n X_\Gamma^{[n]}$ where $X_\Gamma = \widehat{\mathbb{C}^2/\Gamma}$ and $\Gamma = \mathbb{Z}/n\mathbb{Z} \subset SL_2(\mathbb{C})$. This categorified the Fock space. Then in [CLi2], starting with the Heisenberg 2-representation from [CLi1], we defined various complexes of functors giving a 2-representation of $\widehat{\mathfrak{sl}}_n$ which categorified its basic representation. Other representations of $\widehat{\mathfrak{sl}}_n$ can also be categorified this way.

Now, in the story above, one can replace $\widehat{\mathbb{C}^2/\Gamma}$ by the stack $[\mathbb{C}^2/\Gamma]$ and then $\Gamma = \mathbb{Z}/n\mathbb{Z}$ by $\Gamma = \mathbb{C}^\times$ to obtain the stack $[\mathbb{C}^2/\mathbb{C}^\times]$. Then the constructions above generalize to yield categorifications of \mathfrak{sl}_∞ representations. In particular, one can categorify $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$ this way and, using the results in this paper, define all the categorical Reshetikhin-Turaev invariants (of type A) by working with the Hilbert schemes $\bigsqcup_n [\mathbb{C}^2/\mathbb{C}^\times]^{[n]}$.

1.5. Related work on categorical projectors and knot homologies. In the last few years several papers dealing with categorified Jones-Wenzl projectors (and clasps more generally) have appeared. We briefly recall some of these and their relation to our work.

In [Roz] Rozansky categorified the Jones-Wenzl projectors within Bar-Natan's graphical formulation of Khovanov homology [BN1]. In this setup the 1-morphisms are tangles and 2-morphisms are cobordisms modulo certain relations. His construction is essentially the same as ours in that the projectors are defined as infinite braids. However, since he only considers the \mathfrak{sl}_2 case, the braid group actions are simpler and the infinite braids easier to deal with.

Independently, Cooper and Krushkal [CoK] also categorify the Jones-Wenzl projectors within Bar-Natan's setup using a recursive definition of the projectors (incidentally, such recursive descriptions have also been written down for \mathfrak{sl}_3 clasps, see for instance [Kim]). In the case of two strands, they describe the projector explicitly [CoK, Sec. 4.1] as a complex which is very similar to that in (57).

Indeed, if we depict F_2 as a cap and E_2 as a cup then (57) can be identified with the (dual) of their complex. Various follow up computations can be found in [CHK].

One should be able to categorify the \mathfrak{sl}_∞ -module $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes 2}$ using Bar-Natan's setup mentioned above. Having done this, it will not be hard to show that the constructions in this paper agree with the Jones-Wenzl projectors from [Roz, CoK].

In [Ros] Rose works within the Morrison-Nieh [MN] formulation of \mathfrak{sl}_3 knot homology to categorify \mathfrak{sl}_3 clasps. Again, one should be able to categorify $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes 3}$ in this Morrison-Nieh framework. Then the clasps constructed by Rose will be recovered from the construction in this paper.

More generally, one could try to replace tangles with webs and cobordisms with foams. Working with the right setup one should be able to categorify $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$ for any m . The precise relationship between webs and \mathfrak{sl}_∞ (at the decategorified level) will be spelled out in [CKM] using skew Howe duality.

1.5.1. Work of Frenkel-Stroppel-Sussan. Another categorification of Jones-Wenzl projectors appears in [FSS]. The \mathfrak{sl}_2 representation $V^{\otimes n}$ (where V is the standard representation of \mathfrak{sl}_2) is categorified by $\oplus_{k=0}^n A_{k,n} - \text{gmod}$ for some graded algebras $A_{k,n}$ (this follows from [FKS] where these categories correspond to a certain category \mathcal{O}). The Jones-Wenzl projectors are then categorified by explicit $(A_{k,n}, A_{k,n})$ -bimodules.

The simplest nontrivial example of their construction is when $k = 1, n = 2$. In this case $A_{1,2}$ is a 5-dimensional algebra which can be described as the path algebra of the quiver

$$v \bullet \begin{array}{c} \xrightarrow{x} \\ \xleftarrow{y} \end{array} \bullet w \quad \text{with relations} \quad yx = 0.$$

If we denote by e_w the constant path that starts and ends at w then $e_w A_{1,2} e_w \cong \mathbb{C}[t]/t^2$ where $t = xy$. The projector is then defined by the $(A_{1,2}, A_{1,2})$ -bimodule $M_{1,2} := A_{1,2} e_w \otimes_{\mathbb{C}[t]/t^2} e_w A_{1,2}$ (the tensor is derived so this ends up being a complex of bimodules).

On the other hand, [CoK] also defines a complex of bimodules which categorifies this particular Jones-Wenzl projector. Stroppel and Sussan [SS] show that these two projectors are related by Koszul duality.

One can draw the following parallels between their work and the results in this paper.

| | | |
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| $A_{1,2} - \text{gmod}$ | $\langle \sim \sim \sim \sim \sim \rangle$ | coherent sheaves on $T^*\mathbb{P}^1$ (or on its compactification $Y(1,1)$) |
| projector induced by bimodule $M_{1,2}$ | $\langle \sim \sim \sim \sim \sim \rangle$ | functor p^*p_* where $p : T^*\mathbb{P}^1 \rightarrow \overline{T^*\mathbb{P}^1}$ is the affinization map |
| relation proved in [SS] between $M_{1,2}$ and [CoK] projector | $\langle \sim \sim \sim \sim \sim \rangle$ | Proposition 11.8 |
| conjectural relation between [FSS] projectors and [CoK] projectors | $\langle \sim \sim \sim \sim \sim \rangle$ | Conjecture 11.9 |

It is an open problem to generalize the category \mathcal{O} approach of [FSS, SS] from \mathfrak{sl}_2 invariants (and Jones-Wenzl projectors) to \mathfrak{sl}_n invariants (and clasps) and to extend the table above.

1.5.2. Work of Webster. In [W1, W2] Webster also describes a categorification of \mathfrak{sl}_n Reshetikhin-Turaev invariants although his approach is quite different from ours. For each tensor product $V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k}$ of arbitrary (finite dimensional) \mathfrak{sl}_n representations he defines an algebra $A_{\underline{\lambda}}$ generalizing the cyclotomic KLR algebras (a.k.a. Hecke quiver algebras) introduced in [KL1, KL2, KL3, Rou1]. He shows that these algebras have the right Grothendieck group [W1] and then writes down a braid group action on their module categories [W2] which gives rise to homological knot invariants.

It would be interesting to relate his construction with ours. In particular, if $\lambda_k = \Lambda_{i_k}$ are all *fundamental* weights then variety $Y(\underline{i})$ we use in section 8 should be related to the algebra A_λ . Furthermore, there should exist two functors

$$D(A_\lambda\text{-mod}) \longrightarrow D(A_{\sum_k \Lambda_{i_k}}\text{-mod}) \longrightarrow D(A_\lambda\text{-mod})$$

whose composition categorifies the clasp $P\mathbf{1}_{\underline{i}}$. This composition should agree with the clasp $P^{-}\mathbf{1}_{\underline{i}} : D(A_\lambda\text{-mod}) \rightarrow D(A_\lambda\text{-mod})$ which was defined above as an infinite braid.

1.5.3. Work of Wu and Yonezawa. In [Wu] Wu categorifies the \mathfrak{sl}_m invariants of links labeled by fundamental representations using categories of matrix factorizations. This generalizes the approach of Khovanov-Rozansky [KR]. To a crossing he assigns a complex of matrix factorizations and then checks that these complexes satisfy the Reidemeister moves. Independently, Yonezawa [Y] has a very similar construction but does not check the Reidemeister relations in all cases (supposedly because the calculations become very complicated).

One expects that their categories of matrix factorizations assemble to give a categorification of $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ or $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$. The Reidemeister relations would then follow immediately as in our discussion above.

1.6. Finite dimensionality, functoriality and cobordisms. In [CaK1, Sect. 7] we showed that the tangle invariant $\Psi(T)$ we had defined extends to an invariant of tangle cobordisms. This means that for any tangle cobordism $T \rightarrow T'$ there exists a natural transformation $\Psi(T) \rightarrow \Psi(T')$ (defined up to scaling). The model we used involved four generating cobordisms: the birth and death of a circle and two saddles (see [CaK1, Fig. 4]).

The saddles were both defined as adjunction maps since the cap and cup functors are adjoint to each other. The circle is also the composition of a cup and a cap and its death and birth were similarly defined by adjunction maps. In that case the homology of the unknot was isomorphic to $\mathbb{C}[t]/t^2 \cong H^*(\mathbb{P}^1)$ and the cobordisms endowed this vector space with the natural ring structure.

Now, consider the \mathfrak{sl}_m tangle invariants in this paper, obtained from a categorification of $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ as explained in section 1.2. To simplify our discussion suppose all the strands are labeled by a fixed representation V_λ . If $\lambda = \lambda_k$ is a fundamental weight then no clasps are needed and subsequently the homology of any link is finite dimensional.

In this case, the arguments in [CaK1, Sect. 7] translate directly to give cobordism invariants in the sense above. For example, a cup and cap are given by functors $E_i^{(r)}$ and $F_i^{(r)}$ using our categorification of $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ and these functors are (by definition) adjoint to each other. If K_λ is the unknot labeled by V_λ where $\lambda = \Lambda_k$ then its homology $\mathcal{H}_{-}^{*,*}(K_\lambda)$ is isomorphic to $H^*(\mathbb{G}(k, m))$ where $\mathbb{G}(k, m)$ is the Grassmannian. The cobordism structure then endows it with the obvious ring structure.

On the other hand, if λ is not a fundamental weight then one needs to use the clasps P^- which are infinite complexes. Subsequently, the homology of a link will be infinite dimensional. The approach from [CaK1, Sect. 7] still gives us cobordism maps for saddles but we can no longer define maps for birth and death cobordisms.

The resulting homology of the unknot $\mathcal{H}_{-}^{*,*}(K_\lambda)$ is now infinite dimensional (see section 10 for some sample computations). But since we still have saddle cobordisms this endows $\mathcal{H}_{-}^{*,*}(K_\lambda)$ with a ring structure. It would be interesting to identify this ring structure more explicitly.

1.6.1. Cobordisms of Khovanov-Thomas. The definition of a braid group categorification from [CS, KT] involves a further type of cobordism. These cobordisms are between (positive or negative) crossings and the identity functor. There are four maps one needs to define: namely $\text{id} \rightarrow T_i^\pm$ and $T_i^\pm \rightarrow \text{id}$. Here the braid element T_i is given by the complex defined in (4) where $\langle \lambda, \alpha_i \rangle = 0$. In our setup, it suffices to define these four maps when dealing with crossings between two strands labeled by the same *fundamental* representation.

Two of these four maps, namely $\text{id} \rightarrow \mathbb{T}_i$ and $\mathbb{T}_i^{-1} \rightarrow \text{id}$ are easy to define. For instance, \mathbb{T}_i is given by a complex $[\cdots \rightarrow \mathbf{E}_i \mathbf{F}_i \langle -1 \rangle \rightarrow \text{id}]$ and $\text{id} \rightarrow \mathbb{T}_i$ is induced by the identity map $\text{id} \rightarrow \text{id}$. These correspond to maps (3.5) and (3.6) in the example from [KT]. The other two maps involve the natural transformations θ_k (see section 2.2) and correspond to (3.7) and (3.8) in [KT]. The fact that these maps give a cobordism action in the sense of Khovanov-Thomas is not difficult.

The definition in [CS, KT] applies when all strands are labeled the same. There should be a more general definition which works for any labeling of strands. This should include cobordisms $\mathbb{T}_i \leftrightarrow \mathbb{T}_i^{-1}$ between positive and negative crossings of two strands labeled by different weights. In future work, we hope to get back to this question and its relation to constructions in this paper.

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2. STATEMENTS OF RESULTS

2.1. Lie algebras and braid groups. For convenience the base field will always be \mathbb{C} . We deal with representations of $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{g} = \mathfrak{sl}_\infty$. This means that the vertex set I of the Dynkin diagram of \mathfrak{g} is indexed by $\{1, \dots, n-1\}$ if $\mathfrak{g} = \mathfrak{sl}_n$ or by \mathbb{Z} if $\mathfrak{g} = \mathfrak{sl}_\infty$. We fix the following data:

- (i) a free \mathbb{Z} -module X (the weight lattice),
- (ii) for $i \in I$ an element $\alpha_i \in X$ (simple roots),
- (iii) for $i \in I$ an element $\Lambda_i \in X$ (fundamental weights),
- (iv) a symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on X .

These data should satisfy:

- (i) the set $\{\alpha_i\}_{i \in I}$ is linearly independent,
- (ii) for all $i, j \in I$ we have $\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{if } |i - j| > 1 \end{cases}$,
- (iii) $\langle \Lambda_i, \alpha_j \rangle = \delta_{i,j}$ for all $i, j \in I$.

If V is a representation then $V(\mu)$ denotes the weight space corresponding to $\mu \in X$. If $\lambda \in X$ is a dominant weight then V_λ denotes the irreducible representation with highest weight λ . More generally, for a sequence $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$, where each $\lambda_i \in X$ is dominant, $V_{\underline{\lambda}} := V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k}$.

We let $W_{\mathfrak{g}}$ denote the Weyl group of \mathfrak{g} . It has generators s_i for $i \in I$ and relations $s_i^2 = 1$ and

$$\begin{aligned} s_i s_j s_i &= s_j s_i s_j \text{ if } |i - j| = 1, \\ s_i s_j &= s_j s_i \text{ if } |i - j| > 1. \end{aligned}$$

This group acts on X via $s_i \cdot \lambda := \lambda - \langle \lambda, \alpha_i \rangle \alpha_i$.

We let $B_{\mathfrak{g}}$ denote the braid group of type \mathfrak{g} . This has generators σ_i for $i \in I$ and relations

$$\begin{aligned} \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \text{ if } |i - j| > 1. \end{aligned}$$

2.2. Categorical 2-representations. For a review of 2-categories see section 3.1. A *categorical 2-representation of \mathfrak{g}* consists of a graded, additive, \mathbb{C} -linear, idempotent complete 2-category \mathcal{K} where:

- 0-morphisms (objects) are graded, \mathbb{C} -linear, additive categories $\mathcal{D}(\lambda)$ indexed by weights $\lambda \in X$.

- 1-morphisms are functors between these categories and include

$$E_i^{(r)} \mathbf{1}_\lambda : \mathcal{D}(\lambda) \rightarrow \mathcal{D}(\lambda + r\alpha_i) \text{ and } \mathbf{1}_\lambda F_i^{(r)} : \mathcal{D}(\lambda + r\alpha_i) \rightarrow \mathcal{D}(\lambda)$$

where $i \in I$ and $r \in \mathbb{Z}$. By convention $E_i^{(r)} \mathbf{1}_\lambda$ is zero if $r < 0$ and equals $\mathbf{1}_\lambda$ (the identity morphism on $\mathcal{D}(\lambda)$) if $r = 0$.

- 2-morphisms are natural transformations of these functors and include a map $\theta_i : \mathbf{1}_\lambda \rightarrow \mathbf{1}_\lambda \langle 2 \rangle$ for each $i \in I$.

On this data we impose the following conditions.

- (i) (Integrability) For any root $\alpha \in X$ the object $\mathcal{D}(\lambda \pm r\alpha)$ is zero for $r \gg 0$.
- (ii) Each category $\mathcal{D}(\lambda)$ is (split) idempotent complete and the space of homs between any two objects is finite dimensional. Moreover, its Grothendieck group $K(\mathcal{D}(\lambda))$ is finite dimensional.
- (iii) If λ is non-empty then $\text{Hom}_{\mathcal{K}}(\mathbf{1}_\lambda, \mathbf{1}_\lambda \langle l \rangle)$ is zero if $l < 0$ and one-dimensional if $l = 0$. Moreover, the space of 2-morphisms between any two 1-morphisms is finite dimensional.
- (iv) $E_i^{(r)} \mathbf{1}_\lambda$ and $F_i^{(r)} \mathbf{1}_\lambda$ are left and right adjoints of each other up to shift. More precisely
 - (a) $(E_i^{(r)} \mathbf{1}_\lambda)_R \cong \mathbf{1}_\lambda F_i^{(r)} \langle r(\langle \lambda, \alpha_i \rangle + r) \rangle$
 - (b) $(E_i^{(r)} \mathbf{1}_\lambda)_L \cong \mathbf{1}_\lambda F_i^{(r)} \langle -r(\langle \lambda, \alpha_i \rangle + r) \rangle$.
- (v) We have

$$F_i E_i \mathbf{1}_\lambda \cong E_i F_i \mathbf{1}_\lambda \oplus [-\langle \lambda, \alpha_i \rangle] \mathbf{1}_\lambda \text{ if } \langle \lambda, \alpha_i \rangle \leq 0$$

$$E_i F_i \mathbf{1}_\lambda \cong F_i E_i \mathbf{1}_\lambda \oplus [\langle \lambda, \alpha_i \rangle] \mathbf{1}_\lambda \text{ if } \langle \lambda, \alpha_i \rangle \geq 0.$$

- (vi) We have $E_i E_i^{(r)} \mathbf{1}_\lambda \cong \oplus_{[r+1]} E_i^{(r+1)} \mathbf{1}_\lambda \cong E_i^{(r)} E_i \mathbf{1}_\lambda$ and likewise for F s instead of E s.

- (vii) If $|i - j| = 1$ then

$$E_i E_j E_i \mathbf{1}_\lambda \cong E_i^{(2)} E_j \mathbf{1}_\lambda \oplus E_j E_i^{(2)} \mathbf{1}_\lambda$$

while if $|i - j| > 1$ then $E_i E_j \mathbf{1}_\lambda \cong E_j E_i \mathbf{1}_\lambda$.

- (viii) If $i \neq j$ then $F_j E_i \mathbf{1}_\lambda \cong E_i F_j \mathbf{1}_\lambda$.

- (ix) The map $I\theta_j I : E_i \mathbf{1}_\lambda E_i \rightarrow E_i \mathbf{1}_\lambda E_i \langle 2 \rangle$ induces an isomorphism between the summands $E_i^{(2)} \langle 1 \rangle$ on either side if $j = i$ and induces zero otherwise.

Remark 2.1. The word “categorical” refers to the fact that the weight spaces $\mathcal{D}(\lambda)$ are actual categories. This is in contrast to the definition of a “2-representation” from [CLa] where the objects, indexed by weights, have no extra structure.

We will denote by $\text{Kom}^-(\mathcal{D}(\lambda))$ the (bounded above) homotopy category of $\mathcal{D}(\lambda)$. Here the objects are bounded above complexes of objects in $\mathcal{D}(\lambda)$ up to homotopy and 1-morphisms are maps of complexes. Notice that these are now $\mathbb{Z} \oplus \mathbb{Z}$ -graded additive categories since we have the old grading $\langle \cdot \rangle$ and the new cohomological grading which we denote $[\cdot]$.

More generally, we denote by $\text{Kom}^-(\mathcal{K})$ the homotopy 2-category of \mathcal{K} where:

- 0-morphisms (objects) are doubly graded, \mathbb{C} -linear additive categories $\text{Kom}^-(\mathcal{D}(\lambda))$ indexed by weights $\lambda \in X$.
- 1-morphisms are bounded above complexes of 1-morphisms in \mathcal{K} where homotopy equivalent complexes are deemed isomorphic.
- 2-morphisms are maps of complexes made up of 2-morphisms from \mathcal{K} .

In section 3.3 we define certain smaller subcategories $\text{Kom}_*^-(\mathcal{D}(\lambda)) \subset \text{Kom}^-(\mathcal{D}(\lambda))$ and $\text{Kom}_*^-(\mathcal{K}) \subset \text{Kom}^-(\mathcal{K})$ which come equipped with maps

$$p : K(\text{Kom}_*^-(\mathcal{D}(\lambda))) \rightarrow \hat{K}(\mathcal{D}(\lambda)) \text{ and } \tilde{p} : K(\text{Kom}_*^-(\mathcal{K})) \rightarrow \hat{K}(\mathcal{K}).$$

Here $K(\cdot)$ denotes the Grothendieck group of a category and $\hat{K}(\cdot)$ its completion (see section 3.3).

Inside $\text{Kom}_*^-(\mathcal{K})$ one can define the Rickard complexes

$$(4) \quad \mathsf{T}_i \mathbf{1}_\lambda := \cdots \rightarrow \mathsf{E}_i^{(-\langle \lambda, \alpha_i \rangle + s)} \mathsf{F}_i^{(s)} \langle -s \rangle \mathbf{1}_\lambda \xrightarrow{d_i^s} \mathsf{E}_i^{(-\langle \lambda, \alpha_i \rangle + s - 1)} \mathsf{F}_i^{(s-1)} \langle -(s-1) \rangle \mathbf{1}_\lambda \rightarrow \cdots \rightarrow \mathsf{E}_i^{(-\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda$$

$$(5) \quad \mathsf{T}_i \mathbf{1}_\lambda := \cdots \rightarrow \mathsf{F}_i^{(\langle \lambda, \alpha_i \rangle + s)} \mathsf{E}_i^{(s)} \langle -s \rangle \mathbf{1}_\lambda \xrightarrow{d_i^s} \mathsf{F}_i^{(\langle \lambda, \alpha_i \rangle + s - 1)} \mathsf{E}_i^{(s-1)} \langle -(s-1) \rangle \mathbf{1}_\lambda \rightarrow \cdots \rightarrow \mathsf{F}_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda$$

depending on whether $\langle \lambda, \alpha_i \rangle \leq 0$ or $\langle \lambda, \alpha_i \rangle \geq 0$ respectively. Notice that in an integrable representation these complexes are finite since $\mathsf{E}_i^{(s)} \mathbf{1}_\lambda = 0$ and $\mathsf{F}_i^{(s)} \mathbf{1}_\lambda = 0$ for $s \gg 0$. In [CaK3] we showed that these complexes satisfy the braid relations of $B_{\mathfrak{g}}$.

2.3. Statements of results.

2.3.1. *Result #1.* Suppose $\mathfrak{g} = \mathfrak{sl}_n$. Associated to the longest element $\omega \in W_{\mathfrak{g}}$ (with respect to the Bruhat order) we have the complex

$$\mathsf{T}_\omega \mathbf{1}_\lambda : \text{Kom}_*^-(\mathcal{D}(\lambda)) \rightarrow \text{Kom}_*^-(\mathcal{D}(w \cdot \lambda)).$$

For example if $n = 3$ then $I = \{1, 2\}$ and $\mathsf{T}_\omega \mathbf{1}_\lambda = \mathsf{T}_1 \mathsf{T}_2 \mathsf{T}_1 \mathbf{1}_\lambda$. In general,

$$(6) \quad \mathsf{T}_\omega \mathbf{1}_\lambda \cong (\mathsf{T}_{n-1})(\mathsf{T}_{n-2} \mathsf{T}_{n-1}) \cdots (\mathsf{T}_2 \cdots \mathsf{T}_{n-1})(\mathsf{T}_1 \cdots \mathsf{T}_{n-1}) \mathbf{1}_\lambda$$

$$(7) \quad \cong (\mathsf{T}_{n-1} \cdots \mathsf{T}_1)(\mathsf{T}_{n-1} \cdots \mathsf{T}_2) \cdots (\mathsf{T}_{n-1} \mathsf{T}_{n-2})(\mathsf{T}_{n-1}) \mathbf{1}_\lambda.$$

Note that $\mathsf{T}_\omega^2 \mathbf{1}_\lambda = \mathbf{1}_\lambda \mathsf{T}_\omega^2$. Our first Theorem is that the limit $\lim_{\ell \rightarrow \infty} \mathsf{T}_\omega^{2\ell}$ makes sense and defines an idempotent.

Theorem 2.2. *Given a categorical 2-representation \mathcal{K} of \mathfrak{sl}_n there exists a 2-morphism $\mathbf{1}_\lambda \rightarrow \mathsf{T}_\omega^2 \mathbf{1}_\lambda$ so that the limit $\lim_{\ell \rightarrow \infty} \mathsf{T}_\omega^{2\ell} \mathbf{1}_\lambda$ converges to a complex $\mathsf{P}^- \mathbf{1}_\lambda = \mathsf{T}_\omega^\infty \mathbf{1}_\lambda \in \text{Kom}_*^-(\mathcal{K})$ which is an idempotent, meaning that $\mathsf{P}^- \cdot \mathsf{P}^- \mathbf{1}_\lambda \cong \mathsf{P}^- \mathbf{1}_\lambda$.*

2.3.2. *Result #2.* Suppose \mathcal{K} is a categorical 2-representation of $\mathfrak{g} = \mathfrak{sl}_n$ which categorifies the $U_q(\mathfrak{sl}_n)$ -module $\Lambda_q^N(\mathbb{C}^m \otimes \mathbb{C}^n)$ for some m, N . The nonzero weight spaces in this case are parametrized by sequences $\underline{i} = (i_1, \dots, i_n)$ such that $0 \leq i_1, \dots, i_n \leq m$ and $\sum_k i_k = N$. More precisely, the weight space $V(\underline{i})$ is isomorphic to

$$\Lambda_q^{\underline{i}}(\mathbb{C}^m) := \Lambda_q^{i_1}(\mathbb{C}^m) \otimes \cdots \otimes \Lambda_q^{i_n}(\mathbb{C}^m).$$

The composition $\Lambda_q^{\underline{i}}(\mathbb{C}^m) \xrightarrow{\pi} V_{\sum_k \Lambda_{i_k}} \xrightarrow{\iota} \Lambda_q^{\underline{i}}(\mathbb{C}^m)$ (where π and ι are the projection and inclusion maps) defines an idempotent $\mathsf{P} \mathbf{1}_{\underline{i}} := (\iota \circ \pi) \mathbf{1}_{\underline{i}}$. This idempotent is a *generalized Jones-Wenzl projector* or a *clasp* (following the terminology in [Kup]).

Theorem 2.3. *The functor $\mathsf{P}^- \mathbf{1}_{\underline{i}} = \lim_{\ell \rightarrow \infty} \mathsf{T}_\omega^{2\ell} \mathbf{1}_{\underline{i}} \in \text{Kom}_*^-(\mathcal{K})$ categorifies the clasp $\mathsf{P} \mathbf{1}_{\underline{i}}$ in the sense that $\tilde{p}([\mathsf{P}^- \mathbf{1}_{\underline{i}}]) = \mathsf{P} \mathbf{1}_{\underline{i}} \in \hat{K}(\mathcal{K})$.*

2.3.3. *Result #3.* Now fix m and suppose \mathcal{K} is a categorical 2-representation of \mathfrak{sl}_∞ which categorifies the representation $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$. This representation is defined as a limit as $N \rightarrow \infty$ of the $U_q(\mathfrak{sl}_{2N})$ -modules $\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$. As discussed in section 7.1, the nonzero weight spaces of $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ are parametrized by sequences $\underline{i} = (\dots, i_k, i_{k+1}, \dots)$ where $0 \leq i_k \leq m$ and $i_k = 0$ for $k \ll 0$, $i_k = m$ if $k \gg 0$. The weight space $V(\underline{i})$ is isomorphic to

$$\Lambda_q^{\underline{i}}(\mathbb{C}^m) := \cdots \otimes \Lambda_q^{i_k}(\mathbb{C}^m) \otimes \Lambda_q^{i_{k+1}}(\mathbb{C}^m) \otimes \cdots$$

From this data one can construct a categorical (framed) tangle invariant as follows.

- To n strands labeled by fundamental weights $\Lambda_{i_1}, \dots, \Lambda_{i_n}$ one associates the category

$$\text{Kom}_*^-(\mathcal{D}(\underline{0}, i_1, \dots, i_n, \underline{m}))$$

where $\underline{0} = (\dots, 0)$ and $\underline{m} = (m, \dots)$.

- To a (positive) crossing exchanging strands k and $k + 1$ one associates the functor

$$\mathsf{T}_k \mathbf{1}_{\underline{i}} : \mathsf{Kom}_*^-(\mathcal{D}(\underline{i})) \rightarrow \mathsf{Kom}_*^-(\mathcal{D}(s_k \cdot \underline{i})).$$

- To a cap one associates the functor

$$\mathsf{E}_k^{(i_k)} \mathbf{1}_{\underline{i}} : \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots)) \rightarrow \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, 0, m, i_{k+2}, \dots))$$

where $\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, 0, m, i_{k+2}, \dots))$ is identified with $\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_{k+2}, \dots))$ using Corollary 7.8. The cup is defined in a similar way using $\mathbf{1}_{\underline{i}} \mathsf{F}_k^{(i_k)}$ (see section 7.4 for details).

- Finally, to a clasp one associates $\mathsf{P}^- \mathbf{1}_{\underline{i}}$.

Theorem 2.4. *The functors above define categorical invariants of framed tangles labeled by arbitrary representations of \mathfrak{sl}_m . At the level of K -theory these induce the Reshetikhin-Turaev invariants associated to $U_q(\mathfrak{sl}_m)$.*

Remark 2.5. In particular, this means that one can choose isomorphisms

$$K(\mathcal{D}(\underline{0}, i_1, \dots, i_n, \underline{m})) \xrightarrow{\sim} V_{\Lambda_{i_1}} \otimes \dots \otimes V_{\Lambda_{i_n}}$$

which intertwine the maps induced by the functors above for caps, cups and crossings with the Reshetikhin-Turaev maps.

2.3.4. Result #4. In section 8 we define a categorical 2-representation $\mathcal{K}_{\text{Gr}, m}$ which categorifies the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$. This 2-category consists of the following.

- The objects in $\mathcal{K}_{\text{Gr}, m}$ are the derived categories of coherent sheaves on certain twisted products $Y(\underline{i}) = Y(i_1) \tilde{\times} \dots \tilde{\times} Y(i_{2N})$. Here $Y(i_j)$ is the Grassmannian of i_j -planes in \mathbb{C}^m which should be thought of as an orbit in the affine Grassmannian of PGL_m . Then $Y(\underline{i})$ is an iterated Grassmannian bundle which has a concrete description (see section 8.3.1) reminiscent of the cotangent bundle to a flag variety.
- The 1-morphisms in $\mathcal{K}_{\text{Gr}, m}$ are given by kernels

$$\mathcal{E}_k^{(r)} \mathbf{1}_{\underline{i}} \in D(Y(\underline{i}) \times Y(\underline{i} + r\alpha_k)) \text{ and } \mathbf{1}_{\underline{i}} \mathcal{F}_k^{(r)} \in D(Y(\underline{i} + r\alpha_k) \times Y(\underline{i}))$$

as described in section 8.3.3. These kernels are defined by certain correspondences.

- The 2-morphisms θ_i are induced from certain deformations $\tilde{Y}(\underline{i})$ of $Y(\underline{i})$. These deformations are defined in section 8.3.4 while the procedure for obtaining θ_i is discussed in section 8.2.

Theorem 2.6. *The varieties $Y(\underline{i})$, kernels $\mathcal{E}_k^{(r)}$, $\mathcal{F}_k^{(r)}$ and deformations $\tilde{Y}(\lambda)$ define a categorical 2-representation $\mathcal{K}_{\text{Gr}, m}$ of \mathfrak{sl}_∞ . This 2-representation categorifies the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$.*

Remark 2.7. The data consisting of the varieties $Y(\underline{i})$, kernels $\mathcal{E}_k^{(r)}$, $\mathcal{F}_k^{(r)}$ and deformations $\tilde{Y}(\lambda)$ define a geometric categorical \mathfrak{sl}_∞ action in the sense of [CaK3] (see Theorem 8.1).

2.3.5. Result #5. In section 9 we introduce another categorical 2-representation $\mathcal{K}_{\text{Q}, m}$ of \mathfrak{sl}_∞ . This 2-category is defined using coherent sheaves on Nakajima quiver varieties instead of the varieties $Y(\underline{i})$ and categorifies the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$.

Following the same recipe as above one can show that any categorification of $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$ can also be used to categorify the Reshetikhin-Turaev link invariants associated with \mathfrak{sl}_m . The link homologies obtained from $\mathcal{K}_{\text{Gr}, m}$ and $\mathcal{K}_{\text{Q}, m}$ are the same. We explain this geometrically by constructing a restriction 2-functor $\mathcal{K}_{\text{Gr}, m} \rightarrow \mathcal{K}_{\text{Q}, m}$ which categorifies the natural projection of $U_q(\mathfrak{sl}_\infty)$ -modules

$$\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty}) \longrightarrow \Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}.$$

2.3.6. Further Results. The clasp $P^- \in \text{Kom}^-(\mathcal{K}_{\text{Gr},m})$ consists of complexes of 1-morphisms which are bounded above but not below. In section 11 we discuss the analogous projectors $P^+ \in \text{Kom}^+(\mathcal{K}_{\text{Gr},m})$ and explain that they are related to P^- via a duality functor \mathbb{D} .

Given a complex in a triangulated category one can try to define its convolution using an iterated cone construction (see section 11.2). This convolution may not exist and even if it exists it may not be unique. Using that $\mathcal{K}_{\text{Gr},m}$ is also a triangulated 2-category we propose in Conjecture 11.7 that the clasps P^+ can be convolved to obtain a 1-morphism $C(P^+) \in \mathcal{K}_{\text{Gr},m}^-$ (the issue here is convergence).

In Lemma 11.5 we prove this conjecture for the first nontrivial case when $m = 2$. Furthermore, we show that the resulting 1-morphism has a purely geometric description as the pushforward to a singular variety followed by pullback (Proposition 11.8). We conjecture that this geometric description holds for all clasps when $m = 2$ (Conjecture 11.9) but fails when $m > 2$.

Assuming that Conjectures 11.7 and 11.9 are true, we obtain a purely geometric construction via the affine Grassmannian for PGL_2 of all the Reshetikhin-Turaev tangle invariants associated with \mathfrak{sl}_2 . This means that all the functors, including the Jones-Wenzl projectors, have a geometric meaning.

3. SOME PRELIMINARIES

In this section we collect some conventions, definitions and results which will be used later. The reader may skip this and refer to it later as necessary.

Except for some examples in sections 10.3 and 10.4, our ground field will always be \mathbb{C} . We denote by $[n]$ the quantum integer $q^{n-1} + q^{n-3} + \dots + q^{-n+3} + q^{-n+1}$. More generally,

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n] \dots [1]}{([n-k] \dots [1])([k] \dots [1])}.$$

If $f = f_a q^a \in \mathbb{N}[q, q^{-1}]$ and A is an object in a graded category (see below) then we write $\oplus_f A$ for the direct sum $\oplus_{a \in \mathbb{Z}} A^{f_a} \langle a \rangle$. For example, $\oplus_{[n]} A = \oplus_{k=0}^{n-1} A \langle n-1-2k \rangle$.

3.1. Categories. By a graded category \mathcal{C} we mean a category equipped with an auto-equivalence $\langle 1 \rangle$. We denote by $\langle l \rangle$ the auto-equivalence obtained by applying $\langle 1 \rangle$ l times.

3.1.1. Graded 2-categories. A graded additive \mathbb{C} -linear 2-category \mathcal{K} is a category enriched over graded additive \mathbb{C} -linear categories. This means that the Hom categories $\text{Hom}_{\mathcal{K}}(A, B)$ between objects A and B are graded additive \mathbb{C} -linear categories and the composition map $\text{Hom}_{\mathcal{K}}(A, B) \times \text{Hom}_{\mathcal{K}}(B, C) \rightarrow \text{Hom}_{\mathcal{K}}(A, C)$ is a graded additive \mathbb{C} -linear functor.

3.1.2. Idempotent completeness. An additive category \mathcal{C} is idempotent complete if whenever $e \in \text{End}(A)$ and $e^2 = e$ then $A \cong A_1 \oplus A_2$ where e acts by the identity on A_1 and by zero on A_2 . Notice that the derived category of coherent sheaves on any variety is idempotent complete (since the derived category of any abelian category is idempotent complete by Corollary 2.10 of [BS]).

We say that our additive 2-category \mathcal{K} is idempotent complete when the Hom categories $\text{Hom}_{\mathcal{K}}(A, B)$ are idempotent complete for any pair of objects A, B of \mathcal{K} , so that all idempotent 2-morphisms split.

Example 1. Let Y be any proper variety. Then the 2-category where there is a unique object, 1-morphisms are coherent sheaves on $Y \times Y$ (with composition given by convolution) and 2-morphisms are maps of sheaves on $Y \times Y$, is an idempotent complete 2-category.

3.1.3. Triangulated categories. A graded triangulated category is a graded category equipped with a triangulated structure where the autoequivalence $\langle 1 \rangle$ takes exact triangles to exact triangles. The homological shift is denoted $[1]$.

A graded triangulated \mathbb{C} -linear 2-category \mathcal{K}' is a category enriched over graded triangulated \mathbb{C} -linear categories. This means that for any two objects $A, B \in \mathcal{K}'$ the Hom category $\text{Hom}_{\mathcal{K}'}(A, B)$ is a graded additive \mathbb{C} -linear triangulated category. Here are two examples to keep in mind.

Example 2. The homotopy category $\mathcal{K}' := \text{Kom}(\mathcal{K})$ of a graded additive k -linear 2-category \mathcal{K} . The objects of \mathcal{K}' are the same as the objects of \mathcal{K} . The 1-morphisms of \mathcal{K}' are unbounded complexes of 1-morphisms in \mathcal{K} , and 2-morphisms are maps of complexes. Two complexes of 1-morphisms are then deemed isomorphic if they are homotopy equivalent. This makes $\text{Hom}_{\mathcal{K}'}(A, B)$ into a graded triangulated category.

Example 3. The 2-category of Fourier-Mukai (FM) transforms on a collection of varieties (see also section 8.1). The objects are a collection of varieties $\{Y_i\}$. The 1-morphisms are objects in the derived category of coherent sheaves $D(Y_i \times Y_j)$ (these 1-morphisms are known as FM kernels). The 2-morphisms are morphisms between these FM kernels. If all varieties also carry a \mathbb{C}^\times action then the derived categories of \mathbb{C}^\times -equivariant coherent sheaves carry an extra grading and yield a graded triangulated \mathbb{C} -linear 2-category.

3.2. Grothendieck groups. If \mathcal{C} is an idempotent complete, additive category then we can consider the free group generated by isomorphism classes of objects in \mathcal{C} modulo the relation that $[B] = [A] + [C]$ if $B \cong A \oplus C$. If \mathcal{C} is abelian or triangulated we also quotient by this relation if

$$A \rightarrow B \rightarrow C$$

is an exact sequence (or distinguished triangle). By $K(\mathcal{C})$ we denote the tensor product of this quotient with the base field \mathbb{C} , and refer to it as the (split) Grothendieck group of \mathcal{C} .

If \mathcal{C} is also graded then the autoequivalence $\langle -1 \rangle$ corresponds to multiplication by an indeterminate q . More precisely, the object $A\langle -1 \rangle \in \mathcal{C}$ has class $q[A] \in K(\mathcal{C})$. In this case $K(\mathcal{C})$ is a $\mathbb{C}[q, q^{-1}]$ -module rather than just a \mathbb{C} -module.

One can likewise decategorify a 2-category to obtain a 1-category. In the case when \mathcal{K} is a 2-representation of \mathfrak{g} we get the following. The objects are now $K(\mathcal{D}(\lambda))$ which are $\mathbb{C}[q, q^{-1}]$ -modules. The 1-morphisms $E_i^{(r)} \mathbf{1}_\lambda$ and $\mathbf{1}_\lambda F_i^{(r)}$ now induce 1-morphisms

$$E_i^{(r)} \mathbf{1}_\lambda : K(\mathcal{D}(\lambda)) \rightarrow K(\mathcal{D}(\lambda + r\alpha_i)) \text{ and } \mathbf{1}_\lambda F_i^{(r)} : K(\mathcal{D}(\lambda + r\alpha_i)) \rightarrow K(\mathcal{D}(\lambda))$$

of $\mathbb{C}[q, q^{-1}]$ -modules. And we forget about the 2-morphisms altogether. The defining relations in a 2-representation of \mathfrak{g} then imply that these 1-morphisms give $\oplus_\lambda K(\mathcal{D}(\lambda))$ the structure of an integrable $U_q(\mathfrak{g})$ -module.

3.3. The homotopy categories $\text{Kom}_*^-(\mathcal{D}(\lambda))$ and $\text{Kom}_*^-(\mathcal{K})$. Consider the natural map $\mathcal{D}(\lambda) \rightarrow \text{Kom}^-(\mathcal{D}(\lambda))$ given by including in cohomological degree zero. We would like this map to induce an isomorphism on Grothendieck group. Unfortunately, this map is not even injective. To partially fix this problem we will consider a subcategory $\text{Kom}_*^-(\mathcal{D}(\lambda)) \subset \text{Kom}^-(\mathcal{D}(\lambda))$ so that there exist maps

$$\hat{K}(\mathcal{D}(\lambda)) \xrightarrow{i} K(\text{Kom}_*^-(\mathcal{D}(\lambda))) \xrightarrow{p} \hat{K}(\mathcal{D}(\lambda))$$

whose composition is the identity. Here $\hat{K}(\mathcal{D}(\lambda))$ is the completion of $K(\mathcal{D}(\lambda))$ as defined below.

3.3.1. The q -adic norm. Given $f \in \mathbb{C}[q, q^{-1}]$ where $f = \sum_j a_j q^j$ we define

$$|f|_q := q^{-\min\{j: a_j \neq 0\}}.$$

Now fix a basis v_1, \dots, v_m of $K(\mathcal{D}(\lambda))$. This is possible since $K(\mathcal{D}(\lambda))$ is finite dimensional. Given an object $A \in \mathcal{D}(\lambda)$ one can uniquely write its class $[A] \in K(\mathcal{D}(\lambda))$ as $[A] = \sum_{i=1}^m f_i v_i$ where each $f_i \in \mathbb{C}[q, q^{-1}]$. We define

$$|[A]|_q := \sum_i |f_i|_q.$$

Note that this norm depends on the choice of basis v_1, \dots, v_m .

Now we define $\mathbf{Kom}_*^-(\mathcal{D}(\lambda)) \subset \mathbf{Kom}^-(\mathcal{D}(\lambda))$ as the subcategory made up of complexes which are homotopic to a complex $\cdots \rightarrow A^{-2} \rightarrow A^{-1} \rightarrow A^0$ where

$$(8) \quad \lim_{u \rightarrow \infty} \|A^{-u}\|_q = 0$$

where we think of q as a number $q > 1$. In other words, the coefficients of $[A^{-u}] \in K(\mathcal{D}(\lambda))$ contain higher and higher powers of q as $u \rightarrow \infty$.

It is not hard to see that this condition does *not* depend on the choice of basis v_1, \dots, v_m used to define $|\cdot|_q$. It is also clear that this subcategory is closed under taking cones.

Example 4. Let $A \in \mathcal{D}(\lambda)$. Then the complex

$$A^\bullet := [\dots \xrightarrow{0} A\langle -2 \rangle \xrightarrow{0} A\langle -1 \rangle \xrightarrow{0} A]$$

belongs to $\mathbf{Kom}_*^-(\mathcal{D}(\lambda))$. Its class in $K(\mathbf{Kom}_*^-(\mathcal{D}(\lambda)))$ is equal to $\left(\sum_{j \geq 0} (-1)^j q^j\right) [A]$. On the other hand, the complex $[\dots \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} A]$ does not belong to $\mathbf{Kom}_*^-(\mathcal{D}(\lambda))$.

Now let us denote by $\hat{K}(\mathcal{D}(\lambda))$ the completion of $K(\mathcal{D}(\lambda))$ in the q -adic norm. This means that we are allowing elements of the form $f \cdot [A]$ where $f \in \mathbb{C}[[q]][q^{-1}]$ (instead of just $f \in \mathbb{C}[q, q^{-1}]$). Now one can define the map $p : K(\mathbf{Kom}_*^-(\mathcal{D}(\lambda))) \rightarrow \hat{K}(\mathcal{D}(\lambda))$ by taking

$$(9) \quad [A^\bullet] \mapsto \sum_{u=0}^{\infty} (-1)^u [A^{-u}].$$

This sum converges precisely because of condition (8). To see that this map is well defined suppose $A^\bullet \xrightarrow{f} B^\bullet$ is a homotopy equivalence. We need to show that they are mapped to the same thing. Now $\text{Cone}(f)$, which is homotopic to zero, is mapped to $p([A^\bullet]) - p([B^\bullet])$. So it suffice to show that any nul-homotopic complex C^\bullet is mapped to zero.

Now if $C^\bullet = [\dots C^{-2} \rightarrow C^{-1} \xrightarrow{f} C^0]$ is nul-homotopic there is a map $h : C^0 \rightarrow C^{-1}$ so that $f \circ h = \text{id}_{C^0}$. Thus $C^{-1} = C^0 \oplus \hat{C}^{-1}$ for some \hat{C}^{-1} and C^\bullet is homotopic to $[\dots \rightarrow C^{-2} \rightarrow \hat{C}^{-1} \rightarrow 0]$. Now we repeat. Since $\|C^{-u}\|_q \rightarrow 0$ this means that $[C^\bullet]$ has arbitrarily small norm and hence must be zero.

Note that the composition $p \circ i$ is equal to the identity map on $\hat{K}(\mathcal{D}(\lambda))$.

3.3.2. The subcategory $\mathbf{Kom}_*^-(\mathcal{K})$. Suppose $\mathfrak{g} = \mathfrak{sl}_m$. Consider the canonical basis \vec{B} of $\vec{U}_q(\mathfrak{g})$ defined in [Lu]. Since $\oplus_\lambda K(\mathcal{D}(\lambda))$ finite dimensional all but finitely many elements of \vec{B} act by zero (see [Lu] Remark 25.2.4 and Section 23.1.2). So now we can choose a basis and consider the q -adic topology as above.

By definition, a complex belongs to $\mathbf{Kom}_*^-(\mathcal{K})$ if it homotopic to a complex $\cdots \rightarrow A^{-2} \rightarrow A^{-1} \rightarrow A^0$ where $\lim_{u \rightarrow \infty} \|A^{-u}\|_q = 0$. If $\mathfrak{g} = \mathfrak{sl}_\infty$ then a complex belongs to $\mathbf{Kom}_*^-(\mathcal{K})$ if it is homotopic to a complex which belongs to $\mathbf{Kom}_*^-(\mathcal{K}')$ where \mathcal{K}' is the category associated to some $\mathfrak{sl}_m \subset \mathfrak{sl}_\infty$. By construction, a complex in $\mathbf{Kom}_*^-(\mathcal{K})$ acts on $\oplus_\lambda \mathbf{Kom}_*^-(\mathcal{D}(\lambda))$ by preserving $\oplus_\lambda \mathbf{Kom}_*^-(\mathcal{D}(\lambda))$.

All of this allows us to define a map $\tilde{p} : K(\mathbf{Kom}_*^-(\mathcal{K})) \rightarrow \hat{K}(\mathcal{K})$ just like in (9). Here $\hat{K}(\mathcal{K})$ is the q -adic completion of $K(\mathcal{K})$.

3.3.3. The homotopy category $\mathbf{Kom}_*^+(\mathcal{K})$. Instead of considering bounded above complexes one can consider bounded below complexes. This leads to the analogous definitions of $\mathbf{Kom}_*^+(\mathcal{D}(\lambda))$ and $\mathbf{Kom}_*^+(\mathcal{K})$. With the exception of some discussion in section 11 we will work with $\mathbf{Kom}_*^-(\mathcal{K})$. However, we could have worked with $\mathbf{Kom}_*^+(\mathcal{K})$ since all our arguments apply with little or no change.

3.4. Convergence of complexes. We now explain what it means for a sequence of complexes to converge. We try to use the same terminology as in [Roz].

Fix an additive category \mathcal{C} and consider a complex K^\bullet in the homotopy category $\text{Kom}(\mathcal{C})$ of \mathcal{C} . We say that K^\bullet is supported in homological degrees $\leq k$ if it is homotopic to a complex \tilde{K}^\bullet where $\tilde{K}^i = 0$ if $i > k$. The infimum over all such k is called the *homological order* of K^\bullet and is denoted $|K^\bullet|_h$.

A direct system

$$K_\star^\bullet := K_0^\bullet \xrightarrow{f_0} K_1^\bullet \xrightarrow{f_1} K_2^\bullet \xrightarrow{f_2} \dots$$

of complexes in $\text{Kom}^-(\mathcal{C})$ is Cauchy if $\lim_{i \rightarrow \infty} | \text{Cone}(f_i) |_h = -\infty$. Moreover, we say that $\lim_{i \rightarrow \infty} K_\star^\bullet = K^\bullet$ if there exist maps $\tilde{f}_i : K_i^\bullet \rightarrow K^\bullet$ such that \tilde{f}_i is homotopic to $\tilde{f}_{i+1} \circ f_i$ and $\lim_{i \rightarrow \infty} | \text{Cone}(\tilde{f}_i) |_h = -\infty$.

Theorem 3.1. [Roz, Thm. 2.5, 2.6] *A direct system K_\star^\bullet has a limit if and only if it is Cauchy. Moreover, this limit is unique up to homotopy equivalence.*

3.5. Cancellation laws. Suppose that (each graded piece of) the space of homs between two objects in a graded category \mathcal{C} is finite dimensional. Then every object in \mathcal{C} has a unique, up to isomorphism, direct sum decomposition into indecomposables (see section 2.2 of [Rin]).

In this paper we also assume that every graded category \mathcal{C} (such as $\mathcal{D}(\lambda)$) satisfies the following condition:

$$\text{for any nonzero object } A \in \mathcal{C} \text{ we have } A \cong A\langle k \rangle \Rightarrow k = 0.$$

then for any $A, B, C \in \mathcal{C}$ we have the following cancellation laws:

$$A \oplus B \cong A \oplus C \Rightarrow B \cong C \quad A \otimes_{\mathbb{C}} V \cong B \otimes_{\mathbb{C}} V \Rightarrow A \cong B$$

where V is a graded \mathbb{C} vector space. The first law above follows by uniqueness of direct sum decomposition. For a proof of the second fact see section 4.1 of [CaK3].

3.5.1. Gaussian elimination. The following result is a slight generalization of a lemma which Bar-Natan [BN2] calls “Gaussian elimination”. We will use it on several occasions later in the paper.

Lemma 3.2. *Let X, Y, Z, W, U, V be six objects in an additive category and consider a complex*

$$(10) \quad \dots \rightarrow U \xrightarrow{u} X \oplus Y \xrightarrow{f} Z \oplus W \xrightarrow{v} V \rightarrow \dots$$

where $f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and u, v are arbitrary morphisms. If $D : Y \rightarrow W$ is an isomorphism, then (10) is homotopic to a complex

$$(11) \quad \dots \rightarrow U \xrightarrow{u} X \xrightarrow{A-BD^{-1}C} Z \xrightarrow{v|_Z} V \rightarrow \dots$$

Proof. See Lemma 6.2 of [CLi2]. □

3.5.2. The A -rank of a map. Let \mathcal{C} be a graded category and consider an object $A \in \mathcal{C}$ with $\text{End}(A) \cong \mathbb{C}$. Now, suppose X, Y are two objects of \mathcal{C} and let $f : X \rightarrow Y$ be a morphism. Then f gives rise to a bilinear pairing $\text{Hom}(A, X) \times \text{Hom}(Y, A) \rightarrow \text{Hom}(A, A) = \mathbb{C}$. We define the A -rank of f to be the rank of this bilinear pairing and the *total A -rank* of f to be the sum over all $A\langle i \rangle$ -ranks where $i \in \mathbb{Z}$. See section 4.1 of [CaK3] for another discussion of rank.

4. RICKARD COMPLEXES AND RELATIONS

In this section $\mathfrak{g} = \mathfrak{sl}_2$ so we abbreviate E_1 and F_1 as E and F . We will abuse notation slightly and write $\lambda \in \mathbb{Z}$ instead of $\langle \lambda, \alpha_1 \rangle$. Our aim is to define and study how the complexes (4) and (5) commute with functors $E^{(p)}$ and $F^{(p)}$.

4.1. Rickard complexes. First recall the following result which describes how Es and Fs commute.

Lemma 4.1. *We have*

$$\begin{aligned} E^{(a)}F^{(b)}\mathbf{1}_\lambda &\cong \bigoplus_{j \geq 0} \bigoplus_{\substack{\lambda+a-b \\ j}} F^{(b-j)}E^{(a-j)}\mathbf{1}_\lambda \text{ if } \lambda+a-b \geq 0 \\ F^{(b)}E^{(a)}\mathbf{1}_\lambda &\cong \bigoplus_{j \geq 0} \bigoplus_{\substack{-\lambda-a+b \\ j}} E^{(a-j)}F^{(b-j)}\mathbf{1}_\lambda \text{ if } \lambda+a-b \leq 0. \end{aligned}$$

Proof. The proof uses repeatedly the \mathfrak{sl}_2 commutator relation for E and F together with the cancellation laws. See Lemma 4.2 of [CKL3]. \square

Suppose from hereon that $\lambda+r \geq 0$. We define the complex $\mathbf{1}_\lambda \tau_{\lambda+r} \mathbf{1}_{-\lambda-2r} \in \text{Kom}(\mathcal{K})$ by

$$(12) \quad \cdots \rightarrow \mathbf{1}_\lambda E^{(\lambda+r+s)} F^{(s)} \langle -s(r+1) \rangle \xrightarrow{d^s} \mathbf{1}_\lambda E^{(\lambda+r+s-1)} F^{(s-1)} \langle -(s-1)(r+1) \rangle \rightarrow \cdots \rightarrow \mathbf{1}_\lambda E^{(\lambda+r)}$$

$$(13) \quad \cdots \rightarrow \mathbf{1}_\lambda F^{(s)} E^{(\lambda+r+s)} \langle s(r-1) \rangle \xrightarrow{d^s} \mathbf{1}_\lambda F^{(s-1)} E^{(\lambda+r+s-1)} \langle (s-1)(r-1) \rangle \rightarrow \cdots \rightarrow \mathbf{1}_\lambda E^{(\lambda+r)}$$

depending on whether $r \geq 0$ or $r \leq 0$. The differential d^s in (12) is given as the composition

$$\begin{aligned} \mathbf{1}_\lambda E^{(\lambda+r+s)} F^{(s)} &\rightarrow \mathbf{1}_\lambda E^{(\lambda+r+s-1)} E \mathbf{1}_{-\lambda-2r-2s} F F^{(s-1)} \langle -\lambda-r-2s+2 \rangle \\ &\cong \mathbf{1}_\lambda E^{(\lambda+r+s-1)} E E_R \langle \lambda+2r+2s-1 \rangle F^{(s-1)} \langle -\lambda-r-2s+2 \rangle \\ &\rightarrow \mathbf{1}_\lambda E^{(\lambda+r+s-1)} F^{(s-1)} \langle r+1 \rangle \end{aligned}$$

where the first map is inclusion into the lowest degree summand and the last map is adjunction. The differential in (13) is similar.

Likewise, we define the complex $\mathbf{1}_{-\lambda} \tau'_{\lambda+r} \mathbf{1}_{\lambda+2r} \in \text{Kom}(\mathcal{K})$ by

$$(14) \quad \cdots \rightarrow \mathbf{1}_{-\lambda} F^{(\lambda+r+s)} E^{(s)} \langle -s(r+1) \rangle \xrightarrow{d^s} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{(s-1)} \langle -(s-1)(r+1) \rangle \rightarrow \cdots \rightarrow \mathbf{1}_{-\lambda} F^{(\lambda+r)}$$

$$(15) \quad \cdots \rightarrow \mathbf{1}_{-\lambda} E^{(s)} F^{(\lambda+r+s)} \langle s(r-1) \rangle \xrightarrow{d^s} \mathbf{1}_{-\lambda} E^{(s-1)} F^{(\lambda+r+s-1)} \langle (s-1)(r-1) \rangle \rightarrow \cdots \rightarrow \mathbf{1}_{-\lambda} F^{(\lambda+r)}$$

depending on whether $r \geq 0$ or $r \leq 0$. Notice that when $r = 0$ then (12) and (13) are both equal to the complex (4) while (14) and (15) are both equal to (5). In [CR] the complexes (4) and (5) are called Rickard complexes so we will also refer to (12) - (15) as Rickard complexes. Note, however, that if $r \neq 0$ then the complexes (12)–(15) are *not* invertible in the homotopy category.

Remark 4.2. By Lemma 4.3 the space of maps between two consecutive terms in the complex (12) is one-dimensional. This means that any complex which is indecomposable and which has the same terms as the complex in (12) must actually be homotopic to (12). Subsequently, we do not need to worry about what precise differential to choose (just pick any nonzero multiple of the unique map). The same is also true of complexes (13), (14) and (15).

Lemma 4.3. *If $\cdots \rightarrow A \rightarrow A' \rightarrow \cdots$ are two terms appearing in any of the four complexes (12)–(15) then $\text{Hom}(A, A'(\ell))$ and $\text{Hom}(A, A'(\ell))$ are zero if $\ell < 0$ and one-dimensional if $\ell = 0$.*

Proof. The proof is basically the same for all four complexes. We illustrate with (14) where

$$A = \mathbf{1}_{-\lambda} F^{(\lambda+r+s)} E^{(s)} \langle -s(r+1) \rangle \text{ and } A' = \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{(s-1)} \langle -(s-1)(r+1) \rangle.$$

We have:

$$\begin{aligned}
\text{Hom}(\mathbf{A}, \mathbf{A}') &\cong \text{Hom}(\mathbf{1}_{-\lambda} \mathbf{F}^{(\lambda+r+s)} \mathbf{E}^{(s)}, \mathbf{1}_{-\lambda} \mathbf{F}^{(\lambda+r+s-1)} \mathbf{E}^{(s-1)} \langle r+1 \rangle) \\
&\cong \text{Hom}((\mathbf{1}_{-\lambda} \mathbf{F}^{(\mu)})_L \mathbf{F}^{(\mu+1)} \mathbf{E}^{(s)}, \mathbf{E}^{(s-1)} \mathbf{1}_{\lambda+2r} \langle r+1 \rangle) \\
&\cong \text{Hom}(\mathbf{E}^{(\mu)} \mathbf{F}^{(\mu+1)} \mathbf{1}_{\lambda+2r+2s} \mathbf{E}^{(s)} \langle \mu(r+s-1) \rangle, \mathbf{E}^{(s-1)} \langle r+1 \rangle) \\
&\cong \bigoplus_{j=0}^{\mu} \bigoplus_{\left[\begin{smallmatrix} \lambda+2r+2s-1 \\ j \end{smallmatrix} \right]} \text{Hom}(\mathbf{1}_{\lambda+2r+2s-2} \mathbf{F}^{(\mu-j+1)} \mathbf{E}^{(\mu-j)} \mathbf{E}^{(s)}, \mathbf{E}^{(s-1)} \langle r+1-\mu(r+s-1) \rangle) \\
&\cong \bigoplus_{j=0}^{\mu} \bigoplus_{\left[\begin{smallmatrix} \mu+r+s \\ j \end{smallmatrix} \right]} \text{Hom}(\mathbf{E}^{(\mu-j)} \mathbf{E}^{(s)}, \mathbf{E}^{(\mu-j+1)} \mathbf{E}^{(s-1)} \langle t \rangle) \\
&\cong \bigoplus_{j=0}^{\mu} \bigoplus_{\left[\begin{smallmatrix} \mu+r+s \\ j \end{smallmatrix} \right]} \text{Hom} \left(\bigoplus_{\left[\begin{smallmatrix} \mu-j+s \\ s \end{smallmatrix} \right]} \mathbf{E}^{(\mu-j+s)}, \bigoplus_{\left[\begin{smallmatrix} \mu-j+s \\ s-1 \end{smallmatrix} \right]} \mathbf{E}^{(\mu-j+s)} \langle t \rangle \right)
\end{aligned}$$

where, for convenience, we abbreviated $\mu := \lambda + r + s - 1$ and

$$t = r + 1 - \mu(r + s - 1) - (\mu - j + 1)(2\mu + r + s - 1 - j).$$

Thus the last line is a direct sum of terms of the form $\text{Hom}(\mathbf{E}^{(\mu-j+s)}, \mathbf{E}^{(\mu-j+s)} \langle k \rangle)$ for some k . Since $\left[\begin{smallmatrix} N \\ m \end{smallmatrix} \right]$ is supported in degrees $-m(N-m), \dots, m(N-m)$ we find that

$$k \leq j(\mu + r + s - j) + s(\mu - j) + (s-1)(\mu - j + 1) + t = -2(\mu - j)^2 - 2(\mu - j)(r + 1).$$

Since $\mu \geq j$ and $\mathbf{E}^{(\mu-j+s)}$ has no negative degree endomorphisms we see that $\text{Hom}(\mathbf{A}, \mathbf{A}' \langle \ell \rangle) = 0$ if $\ell < 0$. Moreover, if $\ell = 0$ then there is precisely one term which is nonzero (when $j = \mu$) and we get $\text{Hom}(\mathbf{A}, \mathbf{A}') \cong \text{End}(\mathbf{E}^{(s)}) \cong \mathbb{C}$. A similar argument also works with $\text{Hom}(\mathbf{A}, \mathbf{A}' \langle \ell \rangle)$. \square

4.2. The main Proposition. The following is the main result in this section.

Proposition 4.4. *For $p \geq 0$ we have*

- $\tau_{\lambda} \mathbf{1}_{-\lambda} \mathbf{E}^{(p)} \cong \tau_{\lambda+p} \mathbf{1}_{-\lambda-2p} \langle -p(\lambda+p+1) \rangle [p]$ if $\lambda \geq 0$
- $\tau_{\lambda} \mathbf{1}_{-\lambda} \mathbf{F}^{(p)} \cong \tau_{\lambda-p} \mathbf{1}_{-\lambda+2p}$ if $\lambda - p \geq 0$
- $\mathbf{E}^{(p)} \tau_{\lambda} \mathbf{1}_{-\lambda} \cong \tau_{\lambda+p} \mathbf{1}_{-\lambda} \langle -p(\lambda+p+1) \rangle [p]$ if $\lambda \geq 0$
- $\mathbf{F}^{(p)} \tau_{\lambda} \mathbf{1}_{-\lambda} \cong \tau_{\lambda-p} \mathbf{1}_{-\lambda}$ if $\lambda - p \geq 0$.

Similarly, we have

- $\tau'_{\lambda} \mathbf{1}_{\lambda} \mathbf{E}^{(p)} \cong \tau'_{\lambda-p} \mathbf{1}_{\lambda-2p}$ if $\lambda - p \geq 0$
- $\tau'_{\lambda} \mathbf{1}_{\lambda} \mathbf{F}^{(p)} \cong \tau'_{\lambda+p} \mathbf{1}_{\lambda-2p} \langle -p(\lambda+p+1) \rangle [p]$ if $\lambda \geq 0$
- $\mathbf{E}^{(p)} \tau'_{\lambda} \mathbf{1}_{\lambda} \cong \tau'_{\lambda-p} \mathbf{1}_{\lambda}$ if $\lambda - p \geq 0$
- $\mathbf{F}^{(p)} \tau'_{\lambda} \mathbf{1}_{\lambda} \cong \tau'_{\lambda+p} \mathbf{1}_{\lambda} \langle -p(\lambda+p+1) \rangle [p]$ if $\lambda \geq 0$.

Proof. Let us prove that

$$(16) \quad \tau'_{\lambda} \mathbf{1}_{\lambda} \mathbf{F}^{(p)} \cong \tau'_{\lambda+p} \mathbf{1}_{\lambda-2p} \langle -p(\lambda+p+1) \rangle [p].$$

We do this by induction on p . We assume the result for $p \leq r$ and consider $\mathbf{1}_{-\lambda}\tau'_{\lambda+r}\mathbf{F}$. The general term here is

$$\mathbf{1}_{-\lambda}\mathbf{F}^{(\lambda+r+s)}\mathbf{E}^{(s)}\mathbf{F} \cong \left[\bigoplus_{[\lambda+r+s+1]} \mathbf{F}^{(\lambda+r+s+1)}\mathbf{E}^{(s)} \quad \bigoplus_{[\lambda+2r+s+1]} \mathbf{F}^{(\lambda+r+s)}\mathbf{E}^{(s-1)} \right] \langle -s(r+1) \rangle$$

which gives us a complex

$$\dots \rightarrow \left[\bigoplus_{[\lambda+r+s+1]} \mathbf{F}^{(\lambda+r+s+1)}\mathbf{E}^{(s)} \quad \bigoplus_{[\lambda+2r+s+1]} \mathbf{F}^{(\lambda+r+s)}\mathbf{E}^{(s-1)} \right] \langle -s(r+1) \rangle \xrightarrow{\alpha} \left[\bigoplus_{[\lambda+r+s]} \mathbf{F}^{(\lambda+r+s-1)}\mathbf{E}^{(s-1)} \quad \bigoplus_{[\lambda+2r+s]} \mathbf{F}^{(\lambda+r+s-1)}\mathbf{E}^{(s-2)} \right] \langle -(s-1)(r+1) \rangle \rightarrow \dots$$

By Lemma 4.13 the map α induces a surjective map

$$\bigoplus_{[\lambda+2r+s+1]} \mathbf{F}^{(\lambda+r+s)}\mathbf{E}^{(s-1)} \langle -s(r+1) \rangle \rightarrow \bigoplus_{[\lambda+r+s]} \mathbf{F}^{(\lambda+r+s-1)}\mathbf{E}^{(s-1)} \langle -(s-1)(r+1) \rangle.$$

Using the cancellation Lemma 3.2 on all such terms in every degree we end up with a complex

$$(17) \quad \left[\dots \rightarrow \bigoplus_{[r+1]} \mathbf{F}^{(\lambda+r+s-1)}\mathbf{E}^{(s-1)} \langle -(s-1)(r+2) \rangle \rightarrow \bigoplus_{[r+1]} \mathbf{F}^{(\lambda+r+s-2)}\mathbf{E}^{(s-2)} \langle -(s-2)(r+2) \rangle \rightarrow \dots \right] \langle -(\lambda+2r+2) \rangle [1].$$

We would like to show that this is isomorphic to

$$(18) \quad \bigoplus_{[r+1]} \left[\dots \rightarrow \mathbf{F}^{(\lambda+r+s-1)}\mathbf{E}^{(s-1)} \langle -(s-1)(r+2) \rangle \rightarrow \mathbf{F}^{(\lambda+r+s-2)}\mathbf{E}^{(s-2)} \langle -(s-2)(r+2) \rangle \rightarrow \dots \right] \langle -(\lambda+2r+2) \rangle [1]$$

for some sequence of differentials. To do this we use the following trick. Suppose one has a complex of the form

$$\mathbf{B}^\bullet := \dots \oplus_{[r+1]} \mathbf{A}^{-n} \rightarrow \bigoplus_{[r+1]} \mathbf{A}^{-n+1} \rightarrow \dots$$

where $\text{Hom}(\mathbf{A}^{-n}, \mathbf{A}^{-n+1} \langle \ell \rangle) = 0$ if $\ell < 0$ and $\text{Hom}(\mathbf{A}^{-n}, \mathbf{A}^{-n+1})$ is one-dimensional. Then there are two natural maps

$$\iota : \mathbf{C}_1^\bullet \langle r+1 \rangle \rightarrow \mathbf{B}^\bullet \text{ and } \pi : \mathbf{B}^\bullet \rightarrow \mathbf{C}_2^\bullet \langle -(r+1) \rangle$$

where \mathbf{C}_1^\bullet and \mathbf{C}_2^\bullet are both complexes of the form $\dots \rightarrow \mathbf{A}^{-n} \rightarrow \mathbf{A}^{-n+1} \rightarrow \dots$. These maps are just including into and projecting from \mathbf{B}^\bullet . Now suppose further that there exists a map $\Theta : \mathbf{B}^\bullet \rightarrow \mathbf{B}^\bullet \langle 2 \rangle$ which, in every homological degree $-n$, induces an isomorphism between r summands \mathbf{A}^{-n} . Then the composition

$$\mathbf{C}_1^\bullet \xrightarrow{\iota} \mathbf{B}^\bullet \langle -(r+1) \rangle \xrightarrow{\Theta^{r+1}} \mathbf{B}^\bullet \langle r+1 \rangle \xrightarrow{\pi} \mathbf{C}_2^\bullet$$

is a homotopy equivalence. This in turn implies that $\mathbf{B}^\bullet \cong \bigoplus_{[r+1]} \mathbf{C}_1^\bullet$.

Now let us apply this to the case where \mathbf{B}^\bullet is the complex in (17). The Hom conditions on the \mathbf{A}^{-n} s follow from Lemma 4.3. The map Θ is given by $I\Theta I : \tau'_{\lambda+r}\mathbf{1}_{\lambda+2r}\mathbf{F} \rightarrow \tau'_{\lambda+r}\mathbf{1}_{\lambda+2r}\mathbf{F} \langle 2 \rangle$. It has the isomorphism property described above because

$$II\Theta I : \mathbf{F}^{(\lambda+r+s)}\mathbf{E}^{(s)}\mathbf{1}_{\lambda+2r}\mathbf{F} \rightarrow \mathbf{F}^{(\lambda+r+s)}\mathbf{E}^{(s)}\mathbf{1}_{\lambda+2r}\mathbf{F} \langle 2 \rangle$$

induces an isomorphism between all but one summand of the form $\mathbf{F}^{(\lambda+r+s)}\mathbf{E}^{(s-1)}$ on either side (by Corollary 4.12).

Thus, we conclude that $\mathbf{1}_{-\lambda}\tau'_{\lambda+r}\mathbf{F}$ is isomorphic to a complex as in (18) for some choice of differentials. On the other hand, by induction we know that $\mathbf{1}_{-\lambda}\tau'_\lambda\mathbf{F}^{(r)} \cong \mathbf{1}_{-\lambda}\tau'_{\lambda+r}\langle -r(\lambda+r+1) \rangle [r]$, which means

$$\mathbf{1}_{-\lambda}\tau'_{\lambda+r}\mathbf{F} \cong \mathbf{1}_{-\lambda}\tau'_\lambda\mathbf{F}^{(r)}\mathbf{F} \langle r(\lambda+r+1) \rangle [-r] \cong \bigoplus_{[r+1]} \mathbf{1}_{-\lambda}\tau'_\lambda\mathbf{F}^{(r+1)} \langle r(\lambda+r+1) \rangle [-r].$$

Hence $\mathbf{1}_{-\lambda}\tau'_\lambda\mathbf{F}^{(r+1)}$ must be isomorphic to one of the summands in (18), namely, to a complex

$$\left[\dots \rightarrow \mathbf{F}^{(\lambda+r+s-1)}\mathbf{E}^{(s-1)} \langle -(s-1)(r+2) \rangle \rightarrow \mathbf{F}^{(\lambda+r+s-2)}\mathbf{E}^{(s-2)} \langle -(s-2)(r+2) \rangle \rightarrow \dots \right] \langle -(r+1)(\lambda+r+2) \rangle [r+1].$$

Notice that the terms in this complex are the same as those in $\mathbf{1}_{-\lambda}\tau'_{\lambda+r+1}\langle-(r+1)(\lambda+r+2)\rangle[r+1]$. Since $\mathbf{1}_{-\lambda}\tau'_\lambda$ is invertible $\mathbf{1}_{-\lambda}\tau'_\lambda\mathbf{F}^{(r+1)}$ must be indecomposable. Hence, by Remark 4.2,

$$\mathbf{1}_{-\lambda}\tau'_\lambda\mathbf{F}^{(r+1)} \cong \mathbf{1}_{-\lambda}\tau'_{\lambda+r+1}\langle-(r+1)(\lambda+r+2)\rangle[r+1]$$

and our induction is complete. \square

Since $\mathbf{1}_\lambda\tau_\lambda = \mathbf{1}_\lambda\mathbf{T}$ the following is an immediate Corollary of Proposition 4.4.

Corollary 4.5. *For $p \geq 0$ we have*

- $\mathbf{T}\mathbf{1}_\lambda\mathbf{F}^{(p)} \cong \mathbf{1}_{-\lambda}\mathbf{E}^{(p)}\mathbf{T}\langle-p(\lambda+p+1)\rangle[p]$ if $\lambda \geq 0$
- $\mathbf{T}\mathbf{1}_\lambda\mathbf{E}^{(p)} \cong \mathbf{1}_{-\lambda}\mathbf{F}^{(p)}\mathbf{T}\langle-p(-\lambda+p+1)\rangle[p]$ if $\lambda \leq 0$
- $\mathbf{T}\mathbf{1}_\lambda\mathbf{E}^{(\lambda)} \cong \mathbf{1}_{-\lambda}\mathbf{F}^{(\lambda)}\mathbf{T}$ if $\lambda \geq 0$
- $\mathbf{T}\mathbf{1}_\lambda\mathbf{F}^{(-\lambda)} \cong \mathbf{1}_{-\lambda}\mathbf{E}^{(-\lambda)}\mathbf{T}$ if $\lambda \leq 0$.

4.3. The nilHecke algebra: a discussion. To prove Lemma 4.13 below we need to use (in a minor way) the action of the nilHecke on a categorical 2-representation of \mathfrak{sl}_2 . We now discuss this and related results.

In [CKL2] we prove that a geometric categorical \mathfrak{sl}_2 action induces an action of the nilHecke algebra on \mathbf{E}^n (and likewise on \mathbf{F}^n). This nilHecke action consists of two types of maps (natural transformations)

$$X : \mathbf{E}\mathbf{1}_\lambda \rightarrow \mathbf{E}\langle 2 \rangle \mathbf{1}_\lambda \text{ and } T : \mathbf{E}\mathbf{E}\mathbf{1}_\lambda \rightarrow \mathbf{E}\mathbf{E}\langle -2 \rangle \mathbf{1}_\lambda$$

satisfying the following relations

- (i) $T^2 = 0$ where $T \in \text{End}(\mathbf{E}\mathbf{E}\mathbf{1}_\lambda)$,
- (ii) $(IT)(TI)(IT) = (TI)(IT)(TI)$ where $TI, IT \in \text{End}(\mathbf{E}\mathbf{E}\mathbf{E}\mathbf{1}_\lambda)$,
- (iii) $(XI)T - T(IX) = I = -(IX)T + T(XI)$ where $XI, IX, T \in \text{End}(\mathbf{E}\mathbf{E}\mathbf{1}_\lambda)$.

Our definition from section 2.2 is an alternative (less geometric) definition of a geometric categorical \mathfrak{sl}_2 action. The main difference is that, in a geometric categorical action, the 2-morphism θ is encoded by certain deformations of the varieties involved (see section 8.2 for a more detailed explanation).

If you examine the construction of $X : \mathbf{E}\mathbf{1}_\lambda \rightarrow \mathbf{E}\mathbf{1}_\lambda\langle 2 \rangle$ from [CKL2] you find that X is a linear combination

$$(19) \quad a(II\theta) + b(\theta II) : \mathbf{1}_{\lambda+2}\mathbf{E}\mathbf{1}_\lambda \rightarrow \mathbf{1}_{\lambda+2}\mathbf{E}\mathbf{1}_\lambda\langle 2 \rangle$$

for some $a, b \in \mathbb{C}^\times$. All the arguments in [CKL2] work without any change to prove the following.

Theorem 4.6. *Suppose \mathcal{K} is a categorical 2-representation of \mathfrak{sl}_2 . Then there exist 2-morphisms X, T which induce an action of the nilHecke algebra on \mathbf{E}^n and \mathbf{F}^n . The 2-morphism $X : \mathbf{E}\mathbf{1}_\lambda \rightarrow \mathbf{E}\mathbf{1}_\lambda$ is defined as a linear combination $a(II\theta) + b(\theta II)$ for some $a, b \in \mathbb{C}^\times$.*

Remark 4.7. To be precise, X is equal to the linear combination (19) only up to certain endomorphisms of $\mathbf{1}_\lambda$ which we call “transient” maps in [CKL2]. However, these maps have the property that whenever they are sandwiched in the middle of $\mathbf{E}\mathbf{E}$, $\mathbf{F}\mathbf{F}$, $\mathbf{E}\mathbf{F}$ or $\mathbf{F}\mathbf{E}$ they can be moved either to the left or to the right side (which side depends on the weight space). For this reason we can safely ignore them in our discussion.

4.4. Some useful maps. Recall that

$$(20) \quad \mathbf{F}\mathbf{F}^{(r)} \cong \oplus_{[r+1]} \mathbf{F}^{(r+1)} \cong \mathbf{F}^{(r)}\mathbf{F}$$

so (abusing notation a little) we can define maps

$$\iota : \mathbf{F}^{(r+1)} \rightarrow \mathbf{F}\mathbf{F}^{(r)}\langle -r \rangle \text{ and } \iota : \mathbf{F}^{(r+1)} \rightarrow \mathbf{F}^{(r)}\mathbf{F}\langle -r \rangle$$

by including into the lowest summand. Including into the lowest (as opposed to say highest) summand is natural because there is a unique (up to a nonzero multiple) such map *i.e.* the map does not depend

on the choice of isomorphisms in (20). This is a consequence of $\text{Hom}(\mathbf{F}^{(r+1)}, \mathbf{F}^{(r+1)}\langle\ell\rangle)$ being zero if $\ell < 0$ and one-dimensional of $\ell = 0$.

Likewise, there are maps

$$\pi : \mathbf{FF}^{(r)} \rightarrow \mathbf{F}^{(r+1)}\langle -r \rangle \text{ and } \pi : \mathbf{F}^{(r)}\mathbf{F} \rightarrow \mathbf{F}^{(r+1)}\langle -r \rangle$$

given by projecting out of the top degree summand in $\mathbf{FF}^{(r)}$. Again, for the same reason, these maps are unique.

Next we have the adjunction maps

$$\epsilon : \mathbf{FE}\mathbf{1}_\mu \rightarrow \mathbf{1}_\mu\langle\mu+1\rangle \text{ and } \eta : \mathbf{1}_\mu \rightarrow \mathbf{EF}\mathbf{1}_\mu\langle -\mu+1\rangle.$$

Notice that the composition

$$\mathbf{F}\mathbf{1}_\mu \xrightarrow{I\eta} \mathbf{FEF}\mathbf{1}_\mu\langle -\mu+1\rangle \xrightarrow{\epsilon I} \mathbf{F}\mathbf{1}_\mu$$

is the identity (this follows from the basic properties of adjunction maps).

More generally we can define the map ϵ' as the composition

$$\epsilon' : \mathbf{F}\mathbf{1}_\mu \mathbf{E}^{(s)} \xrightarrow{I\iota} \mathbf{F}\mathbf{1}_\mu \mathbf{EE}^{(s-1)}\langle -(s-1)\rangle \xrightarrow{\epsilon} \mathbf{E}^{(s-1)}\langle \mu-s\rangle.$$

This map is adjoint to

$$\mathbf{1}_\mu \mathbf{E}^{(s)} \rightarrow (\mathbf{1}_{\mu-2}\mathbf{F})_R \mathbf{E}^{(s-1)}\langle \mu-s\rangle \cong \mathbf{1}_\mu \mathbf{EE}^{(s-1)}\langle -s+1\rangle$$

which must be the map ι . Thus ϵ' is also unique. Likewise, we define η' as the composition

$$\eta' : \mathbf{E}^{(s-1)}\mathbf{1}_{\mu+2} \xrightarrow{I\eta} \mathbf{E}^{(s-1)}\mathbf{1}_\mu \mathbf{EF}\langle -\mu-1\rangle \xrightarrow{\pi I} \mathbf{E}^{(s)}\mathbf{F}\langle -\mu-s\rangle$$

which is also unique. We denote by χ the composition

$$\chi : \mathbf{FE}^{(s-1)}\mathbf{1}_{\mu+2} \xrightarrow{I\eta'} \mathbf{F}\mathbf{1}_{\mu+2s} \mathbf{E}^{(s)}\mathbf{F}\langle -\mu-s\rangle \xrightarrow{\epsilon' I} \mathbf{E}^{(s-1)}\mathbf{F}\mathbf{1}_{\mu+2}.$$

Note that χ is adjoint to the composition

$$\mathbf{E}^{(s-1)}\mathbf{E} \xrightarrow{\pi} \mathbf{E}^{(s)}\langle -s+1\rangle \xrightarrow{\iota} \mathbf{EE}^{(s-1)}\langle -2s+2\rangle$$

which is nonzero. Thus $\chi \neq 0$. On the other hand, assuming $\mu+2 \geq 0$, we have

$$\mathbf{E}^{(s-1)}\mathbf{F}\mathbf{1}_{\mu+2} \cong \mathbf{FE}^{(s-1)}\mathbf{1}_{\mu+2} \oplus_{[\mu+s]} \mathbf{E}^{(s-2)}\mathbf{1}_{\mu+2}$$

and one can check using this isomorphism that $\text{Hom}(\mathbf{FE}^{(s-1)}\mathbf{1}_{\mu+2}, \mathbf{E}^{(s-1)}\mathbf{F}\mathbf{1}_{\mu+2})$ is one-dimensional. This means that χ is just the inclusion of $\mathbf{FE}^{(s-1)}$ into $\mathbf{E}^{(s-1)}\mathbf{F}$.

Lemma 4.8. *The map $\chi : \mathbf{FE}\mathbf{1}_\mu \rightarrow \mathbf{EF}\mathbf{1}_\mu$ is equal to the composition*

$$\mathbf{FE}\mathbf{1}_\mu \xrightarrow{\eta II} \mathbf{EFFE}\langle -\mu+1\rangle \mathbf{1}_\mu \xrightarrow{IT I} \mathbf{EFFE}\langle -\mu-1\rangle \mathbf{1}_\mu \xrightarrow{II\epsilon} \mathbf{EF}\mathbf{1}_\mu$$

up to a nonzero multiple.

Proof. By adjunction we have

$$\text{Hom}(\mathbf{FE}\mathbf{1}_\mu, \mathbf{EF}\mathbf{1}_\mu) \cong \text{Hom}(\mathbf{E}_L \mathbf{F}\mathbf{1}_{\mu+2}, \mathbf{FE}_L \mathbf{1}_{\mu+2}) \cong \text{Hom}(\mathbf{FF}\mathbf{1}_{\mu+2}, \mathbf{FF}\mathbf{1}_{\mu+2}\langle -2\rangle) \cong \mathbb{C}.$$

So it suffices to show that the composition above is nonzero. On the other hand, the composition above is adjoint to the nonzero map $T : \mathbf{FF} \rightarrow \mathbf{FF}\langle -2\rangle$ and hence is nonzero. \square

Corollary 4.9. *The composition $\mathbf{FE}\mathbf{1}_\mu \xrightarrow{\chi} \mathbf{EF}\mathbf{1}_\mu \xrightarrow{IX} \mathbf{EF}\langle 2\rangle \mathbf{1}_\mu$ is equal to a nonzero linear combination of the compositions $\mathbf{FE}\mathbf{1}_\mu \xrightarrow{\epsilon} \mathbf{1}_\mu\langle\mu+1\rangle \xrightarrow{\eta} \mathbf{EF}\langle 2\rangle \mathbf{1}_\mu$ and $\mathbf{FE}\mathbf{1}_\mu \xrightarrow{XI} \mathbf{FE}\langle 2\rangle \mathbf{1}_\mu \xrightarrow{\chi} \mathbf{EF}\langle 2\rangle \mathbf{1}_\mu$.*

Proof. This follows by using the description of χ from Lemma 4.8 and then applying the nilHecke relation (iii). \square

Lemma 4.10. *The total $F^{(r+1)}$ -rank of the maps $F^{(r)}F1_\lambda \xrightarrow{IX} F^{(r)}F1_\lambda$ and $FF^{(r)}1_\lambda \xrightarrow{XI} FF^{(r)}1_\lambda$ is μ (and similarly if we replace the F s by E s).*

Proof. When $r = 1$ this is essentially condition (ix). More generally this follows from Proposition 4.2 of [CKL2]. \square

Lemma 4.11. *If $\mu + s - 1 \geq 0$ then the total $E^{(s-1)}$ -rank of $E^{(s)}F1_\mu \xrightarrow{IX} E^{(s)}F1_\mu$ is $\mu + s - 2$.*

Proof. We prove this by induction on s . If $\mu < 0$ then the base case is $s = -\mu + 1$, otherwise it is $s = 1$. For convenience, let us assume $\mu \geq 0$ so that the base case is $s = 1$. We need to show that the total 1 -rank of $IX : EF1_\mu \rightarrow EF1_\mu$ is $\mu - 1$. To do this we use a trick. Consider instead the map

$$(21) \quad EFF1_\mu \xrightarrow{IIX} EFF\langle 2 \rangle 1_\mu.$$

Now $EFF1_\mu \cong \oplus_{[2]} EF^{(2)}1_\mu$ and by Lemma 4.10 the map IIX induces an isomorphism between summands $EF^{(2)}\langle 1 \rangle 1_\mu$ on either side. Since $EF^{(2)}1_\mu \cong F^{(2)}E1_\mu \oplus_{[\mu-1]} F1_\mu$ this means that the total F -rank of (21) is (at least) $\mu - 1$.

On the other hand, $EFF1_\mu \cong FEF1_\mu \oplus_{[\mu-2]} F1_\mu$. The map in (21) restricts to X on the summands F and to IIX on $FEF1_\mu$. Thus the total F -rank of (21), which is at least $\mu - 1$, is the same as the total 1 -rank of $IX : EF1_\mu \rightarrow EF1_\mu$ (this proves the base case).

For the induction step consider

$$(22) \quad IIX : E^{(s)}EF1_\mu \rightarrow E^{(s)}EF\langle 2 \rangle 1_\mu.$$

It suffices to show that the total $E^{(s)}$ -rank of this map is at least $(s+1)(\mu + s - 2) + 1$ because then the total $E^{(s)}$ -rank of $IX : E^{(s+1)}F1_\mu \rightarrow E^{(s+1)}F\langle 2 \rangle 1_\mu$ must be at least $(\mu + s - 1)$ which proves the induction step.

Now let us compute the total $E^{(s)}$ -rank of (22) in a different way. Note that $E^{(s)}EF1_\mu \cong E^{(s)}FE1_\mu \oplus_{[\mu]} E^{(s)}1_\mu$. First $IX : EF1_\mu \rightarrow EF\langle 2 \rangle 1_\mu$ has total 1 -rank $(\mu - 1)$ and hence induces a map $\oplus_{[\mu]} E^{(s)}1_\mu \rightarrow \oplus_{[\mu]} E^{(s)}\langle 2 \rangle 1_\mu$ with total $E^{(s)}$ -rank $(\mu - 1)$.

Secondly, the map induced by (22) on the summand $E^{(s)}FE1_\mu$ is the composition of

$$E^{(s)}FE1_\mu \xrightarrow{IX} E^{(s)}EF1_\mu \xrightarrow{IIX} E^{(s)}EF\langle 2 \rangle 1_\mu$$

with the projection back to the $E^{(s)}FE1_\mu$. By Corollary 4.9, this composition is a linear combination of the two compositions

$$\begin{aligned} E^{(s)}FE1_\mu &\xrightarrow{IXI} E^{(s)}FE\langle 2 \rangle 1_\mu \xrightarrow{IX} E^{(s)}EF\langle 2 \rangle 1_\mu \\ E^{(s)}FE1_\mu &\xrightarrow{I\epsilon} E^{(s)}\langle \mu + 1 \rangle 1_\mu \xrightarrow{I\eta} E^{(s)}EF\langle 2 \rangle 1_\mu. \end{aligned}$$

The second map has total $E^{(s)}$ -rank zero and, by the induction hypothesis, the first map has total $E^{(s)}$ -rank $s(\mu + s)$.

Finally, the induced map $E^{(s)}FE1_\mu \rightarrow \oplus_{[\mu]} E^{(s)}$ has total $E^{(s)}$ -rank zero simply because it is of the form $E^{(s)}(FE1_\mu \rightarrow 1_\mu)$. Thus we conclude that (22) has total $E^{(s)}$ -rank $(\mu - 1) + s(\mu + s) \geq (s+1)(\mu + s - 2) + 1$ and we are done. \square

Corollary 4.12. *If $\mu + s - 1 \geq 0$ then the total $E^{(s-1)}$ -rank of $E^{(s)}1_{\mu-2}F \xrightarrow{I\theta I} E^{(s)}1_{\mu-2}F$ is $\mu + s - 2$.*

Proof. As discussed in section 4.3 the map $X : F1_\mu \rightarrow F1_\mu$ is a linear combination of $I\theta$ and θI . Now

$$II\theta : E^{(s)}F1_\mu \rightarrow E^{(s)}F\langle 2 \rangle 1_\mu$$

has total $E^{(s-1)}$ -rank zero. So $I\theta I$ and $IX : E^{(s)}F1_\mu \rightarrow E^{(s)}F\langle 2 \rangle 1_\mu$ have the same total $E^{(s-1)}$ -rank. The result now follows by Lemma 4.11. \square

4.5. The main Lemma.

Lemma 4.13. *If $\lambda, r, s \geq 0$ then the map*

$$\mathbf{1}_{-\lambda} F^{(\lambda+r+s)} E^{(s)} F \xrightarrow{d^s I} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{(s-1)} F \langle r+1 \rangle$$

is surjective on summands of the form $\mathbf{1}_{-\lambda} F^{(\lambda+r+s)} E^{(s-1)}$.

Proof. We could try to prove this result by making heavy use of the nilHecke algebra and the machinery developed in [KLMS]. However, to keep things self contained we use a lowbrow approach.

Let $A := \mathbf{1}_{-\lambda} F^{(\lambda+r+s)} E^{(s-1)}$. Note that by Lemma 4.3 we have $\text{End}(A) \cong \mathbb{C}$. By Lemma 4.11, the summands A inside $\mathbf{1}_{-\lambda} F^{(\lambda+r+s)} E^{(s)} F$ are all picked up by the composition

$$(23) \quad A \langle \lambda + 2r + s - 2k \rangle \xrightarrow{\eta} F^{(\lambda+r+s)} E^{(s)} \mathbf{1}_{\lambda+2r} F \langle -2k \rangle \xrightarrow{IIX^k} F^{(\lambda+r+s)} E^{(s)} \mathbf{1}_{\lambda+2r} F$$

where $k = 0, \dots, \lambda + 2r + s$. In order to prove surjectivity of $d^s I$ it suffices to show that the composition of the map in (23) with d^s has A-rank one for $k = 0, \dots, \lambda + r + s$. Now consider the following diagram where $k \in \{0, \dots, \lambda + r + s\}$:

$$(24) \quad \begin{array}{ccccc} F^{(\lambda+r+s)} E^{(s)} F & \xrightarrow{\iota II} & F^{(\lambda+r+s-1)} F E^{(s)} F & \xrightarrow{\epsilon'} & F^{(\lambda+r+s-1)} E^{(s-1)} F \\ \uparrow IIX^k & & \uparrow IIX^k & & \uparrow IIX^k \\ F^{(\lambda+r+s)} E^{(s)} F & \xrightarrow{\iota II} & F^{(\lambda+r+s-1)} F E^{(s)} F & \xrightarrow{I\epsilon' I} & F^{(\lambda+r+s-1)} E^{(s-1)} F \\ \uparrow I\eta' & & \uparrow II\eta' & & \uparrow I\chi \\ F^{(\lambda+r+s)} E^{(s-1)} & \xrightarrow{\iota I} & F^{(\lambda+r+s-1)} F E^{(s-1)} & \xrightarrow{=} & F^{(\lambda+r+s-1)} F E^{(s-1)} \end{array}$$

where, for simplicity, we have omitted the $\langle \cdot \rangle$ shifts. Starting at the bottom left corner and going along the left side and then the top gives the map in (23) composed with d^s . Now the left hand squares and the top right square clearly commute. The bottom right square commutes because of the definition of χ (section 4.4). Thus it suffices to show that the composition along the bottom and then up has A-rank one for $k = 0, \dots, \lambda + r + s$.

This would follow if we can show that the composition

$$\mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} F E^{(s-1)} \xrightarrow{I\chi} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{(s-1)} F \xrightarrow{IIX} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{(s-1)} F \langle 2 \rangle$$

has A-rank $(\lambda + r + s - 1)$. Equivalently, it suffices to show that

$$(25) \quad \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} F E^{s-1} \xrightarrow{I\chi'} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-1} F \xrightarrow{IIX} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-1} F \langle 2 \rangle$$

has A-rank $(\lambda + r + s - 1)(s - 1)!$ where χ' here is the composition

$$F E^{s-1} \xrightarrow{\chi I^{s-2}} E F E^{s-2} \rightarrow \dots \rightarrow E^{s-2} F E \xrightarrow{I^{s-2} \chi} E^{s-1} F.$$

Now, using Corollary 4.9, the composition

$$\mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} F E \xrightarrow{II^{s-2} \chi} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} E F \xrightarrow{II^{s-1} X} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} E F \langle 2 \rangle$$

is a linear combination of the composition

$$(26) \quad \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} F E \xrightarrow{II^{s-2} \epsilon} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} \langle \lambda + 2r + 3 \rangle \xrightarrow{II^{s-2} \eta} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} E F \langle 2 \rangle$$

and the composition

$$(27) \quad \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} F E \xrightarrow{II^{s-2} \chi'} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} E F \xrightarrow{II^{s-2} X I} \mathbf{1}_{-\lambda} F^{(\lambda+r+s-1)} E^{s-2} E F \langle 2 \rangle.$$

The composition in (26) factors through $\mathbf{1}_{-\lambda}F^{(\lambda+r+s-1)}E^{s-2}$ which contains no summands A . This means that the A -rank of the composition in (26) is zero and hence the A -rank of (25) and (27) are the same. Repeating this argument we find that the A -rank of (25) is the same as the A -rank of the composition

$$\mathbf{1}_{-\lambda}F^{(\lambda+r+s-1)}FE^{s-1} \xrightarrow{IXI^{s-1}} \mathbf{1}_{-\lambda}F^{(\lambda+r+s-1)}FE^{s-1}\langle 2 \rangle \xrightarrow{IX'} F^{(\lambda+r+s-1)}E^{s-1}F\langle 2 \rangle.$$

Since χ' is an inclusion, the A -rank of this composition is the same as that of

$$(28) \quad \mathbf{1}_{-\lambda}F^{(\lambda+r+s-1)}FE^{s-1} \xrightarrow{IXI^{s-1}} \mathbf{1}_{-\lambda}F^{(\lambda+r+s-1)}FE^{s-1}\langle 2 \rangle.$$

By Lemma 4.10 the map $\mathbf{1}_{-\lambda}F^{(\lambda+r+s-1)}F \xrightarrow{IX} \mathbf{1}_{-\lambda}F^{(\lambda+r+s-1)}F\langle 2 \rangle$ has $F^{(\lambda+r+s)}$ -rank $(\lambda + r + s)$. This means that the A -rank of (28) is $(\lambda + r + s - 1)(s - 1)!$ and we are done. \square

5. THE FUNCTOR $P^- = T_\omega^\infty$

In this section we prove that $\lim_{\ell \rightarrow \infty} T_\omega^{2\ell}$ is well defined and that it belongs to $\text{Kom}_*^-(\mathcal{K})$ (Theorem 2.2).

5.1. Braid group actions. In [CaK3] we showed that a geometric categorical \mathfrak{sl}_n action induces an action of the braid group on its weight spaces via the complexes $T_i \mathbf{1}_\lambda$. The definition of a geometric action was tailored to work with categories of coherent sheaves. However, the same proof works to show the following.

Proposition 5.1. *Suppose \mathcal{K} is a categorical 2-representation of \mathfrak{sl}_n , then the complexes $T_i \mathbf{1}_\lambda$ satisfy the braid relations $T_i T_j T_i \mathbf{1}_\lambda \cong T_j T_i T_j \mathbf{1}_\lambda$ if $|i - j| = 1$ and $T_i T_j \mathbf{1}_\lambda \cong T_j T_i \mathbf{1}_\lambda$ if $|i - j| > 1$.*

Proof. In [CaK3, Sec. 6] we defined what it means to have a “strong categorical \mathfrak{sl}_n action” and argued that in this case the complexes T_i also induce a braid group action. The difference between this definition and the one of “categorical 2-representations” from this paper is quite small.

First, in [CaK3] one has the nilHecke algebra acting on E_i^k whereas here we only have the map θ_i . However, as mentioned in section 4.3, the map θ_i together with condition (ix) from section 2.2 induces an action of the nilHecke algebra (this is proved in [CKL2]).

Secondly, in [CaK3, Sec.6] one has an additional map $T_{ij} : E_i E_j \mathbf{1}_\lambda \rightarrow E_j E_i \mathbf{1}_\lambda$ whenever $|i - j| = 1$ and this map satisfies the relation

$$(29) \quad T_{ji} T_{ij} = X_i I + I X_j \in \text{End}(E_i E_j \mathbf{1}_\lambda).$$

Let us explain why this extra data is (for our purposes) redundant.

Let us work in \mathcal{K} . By adjunction one can check that $\text{Hom}(E_i E_j \mathbf{1}_\lambda, E_j E_i \mathbf{1}_\lambda \langle 1 \rangle)$ is one-dimensional. Hence the map T_{ij} basically comes for free. On the other hand, relation (29) is only used in one place in the proof of the braid relation. Namely, it is used to show in [CaK3, Lemma 4.9] that the map

$$E_i E_j E_i \langle -1 \rangle \mathbf{1}_\lambda \xrightarrow{T_{ij} I} E_j E_i E_i \mathbf{1}_\lambda$$

induces an isomorphism between the summands $E_j E_i^{(2)} \langle -1 \rangle \mathbf{1}_\lambda$ on either side. However, it turns out one can check this directly as follows.

First note that this is equivalent to showing that the composition

$$(30) \quad E_i E_j E_i \langle -1 \rangle \mathbf{1}_\lambda \xrightarrow{T_{ij} I} E_j E_i E_i \mathbf{1}_\lambda \xrightarrow{IT} E_j E_i E_i \langle -2 \rangle \mathbf{1}_\lambda$$

induces an isomorphism between the summands $E_j E_i^{(2)} \langle -1 \rangle \mathbf{1}_\lambda$ on either side. But, using adjunction, one can check that

$$\text{Hom}(E_i E_j E_i \langle -1 \rangle \mathbf{1}_\lambda, E_j E_i E_i \langle -2 \rangle \mathbf{1}_\lambda) \cong \mathbb{C}.$$

So it suffices to show that this map is nonzero.

Now suppose the composition (30) is zero. Composing it with $IXI : E_j E_i E_i \langle -2 \rangle \mathbf{1}_\lambda \rightarrow E_j E_i E_i \mathbf{1}_\lambda$ and using the nilHecke relation we get

$$(IXI)(IT)(T_{ij}I) = T_{ij}I + (IT)(IIX)(T_{ij}I) = T_{ij}I + (IT)(T_{ij}I)(IIX).$$

So if (30) is zero (meaning $(IT)(T_{ij}I) = 0$) then $T_{ij}I = 0$ which is absurd.

This means that for the purposes of constructing a braid group action, condition (29) from [CaK3] is not necessary. Since all the other conditions in [CaK3, Sec. 6] also appear in the definition of a categorical 2-representation from this paper, the result follows. \square

One of the useful results used in [CaK3] is the following Lemma. We will also use it here on several occasions.

Lemma 5.2. *Suppose $\mathfrak{g} = \mathfrak{sl}_n$ and fix $p \geq 0$. Then*

- $T_j T_i F_j^{(p)} \mathbf{1}_\lambda \cong F_i^{(p)} T_j T_i \mathbf{1}_\lambda$ and $T_j T_i E_j^{(p)} \mathbf{1}_\lambda \cong E_i^{(p)} T_j T_i \mathbf{1}_\lambda$ if $|i - j| = 1$
- $T_i F_j^{(p)} \mathbf{1}_\lambda \cong F_j^{(p)} T_i \mathbf{1}_\lambda$ and $T_i E_j^{(p)} \mathbf{1}_\lambda \cong E_j^{(p)} T_i \mathbf{1}_\lambda$ if $|i - j| > 1$.

Proof. The first assertion follows from Corollary 5.5 of [CaK3]. There we show that $F_{ij}^{(p)} T_i \mathbf{1}_\lambda \cong T_i F_j^{(p)} \mathbf{1}_\lambda$ where F_{ij} is the two term complex $T_{ij} : F_i F_j \langle -1 \rangle \rightarrow F_j F_i$ and $F_{ij}^{(p)}$ is the associated divided power (here $T_{ij} \in \text{Hom}(F_i F_j \langle -1 \rangle, F_j F_i)$ is the unique nonzero map).

Likewise one can show that $T_j F_{ij}^{(p)} \mathbf{1}_\lambda \cong F_i^{(p)} T_j \mathbf{1}_\lambda$. Hence we get

$$T_j T_i F_j^{(p)} \mathbf{1}_\lambda \cong T_j F_{ij}^{(p)} T_i \cong F_i^{(p)} T_j T_i \mathbf{1}_\lambda.$$

The second assertion is clear because E_i and F_i commute with F_j if $|i - j| > 1$. \square

5.2. The map $\mathbf{1} \rightarrow T_\omega^2$. By the definition of $T_i \mathbf{1}_\lambda$ we have natural maps

$$\begin{aligned} F_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda &\longrightarrow T_i \mathbf{1}_\lambda \text{ if } \langle \lambda, \alpha_i \rangle \geq 0 \\ E_i^{(-\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda &\longrightarrow T_i \mathbf{1}_\lambda \text{ if } \langle \lambda, \alpha_i \rangle \leq 0. \end{aligned}$$

Moreover, if $\langle \lambda, \alpha_i \rangle \geq 0$, we have

$$(31) \quad E_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_{s_i \cdot \lambda} F_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda \cong \bigoplus_{j \geq 0} \bigoplus_{\substack{\langle \lambda, \alpha_i \rangle \\ j}} F_i^{(\langle \lambda, \alpha_i \rangle - j)} E_i^{(\langle \lambda, \alpha_i \rangle - j)} \mathbf{1}_\lambda$$

which means that there is a natural map $\mathbf{1}_\lambda \rightarrow E_i^{(\langle \lambda, \alpha_i \rangle)} F_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda$ corresponding to the inclusion of the unique summand $\mathbf{1}_\lambda$ (which occurs when $j = \langle \lambda, \alpha_i \rangle$ in the summation above).

The composition gives us a map $\mathbf{1}_\lambda \rightarrow T_i^2 \mathbf{1}_\lambda$. If $\langle \lambda, \alpha_i \rangle \leq 0$ then the same argument (with the roles of E s and F s switched) also gives such a map.

Now using (6) we have

$$(32) \quad T_\omega^2 \cong [(T_{n-1})(T_{n-2} T_{n-1}) \dots (T_1 \dots T_{n-1})][(T_{n-1} \dots T_1) \dots (T_{n-1} T_{n-2})(T_{n-1})].$$

So repeatedly using the maps $\mathbf{1}_\lambda \rightarrow T_i^2 \mathbf{1}_\lambda$ defined above we obtain a map $\mathbf{1}_\lambda \rightarrow T_\omega^2 \mathbf{1}_\lambda$.

5.3. Convergence of $\lim_{\ell \rightarrow \infty} T_\omega^{2\ell}$.

Proposition 5.3. *Suppose $\mathfrak{g} = \mathfrak{sl}_n$ and fix $p \geq 0$. Then we have*

$$\begin{aligned} T_\omega \mathbf{1}_\lambda F_i^{(p)} &\cong E_{n-i}^{(p)} T_\omega \mathbf{1}_{\lambda + p \alpha_i} \langle -p(\langle \lambda, \alpha_i \rangle + p + 1) \rangle [p] & \text{if } \langle \lambda, \alpha_i \rangle \geq 0 \\ T_\omega \mathbf{1}_\lambda E_i^{(p)} &\cong F_{n-i}^{(p)} T_\omega \mathbf{1}_{\lambda - p \alpha_i} \langle -p(-\langle \lambda, \alpha_i \rangle + p + 1) \rangle [p] & \text{if } \langle \lambda, \alpha_i \rangle \leq 0. \end{aligned}$$

Proof. We have

$$\begin{aligned}
T_\omega F_i^{(p)} \mathbf{1}_{\lambda+p\alpha_i} &\cong (T_{n-1})(T_{n-2}T_{n-1}) \dots (T_2 \dots T_{n-1})(T_1 \dots T_{n-1}) F_i^{(p)} \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong (T_{n-1})(T_{n-2}T_{n-1}) \dots (T_2 \dots T_{n-1}) T_1 \dots T_i T_{i+1} F_i^{(p)} T_{i+2} \dots T_{n-1} \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong (T_{n-1})(T_{n-2}T_{n-1}) \dots (T_2 \dots T_{n-1}) T_1 \dots T_{i-1} F_{i+1}^{(p)} T_i T_{i+1} \dots T_{n-1} \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong (T_{n-1})(T_{n-2}T_{n-1}) \dots (T_2 \dots T_{n-1}) F_{i+1}^{(p)} (T_1 \dots T_{n-1}) \mathbf{1}_{\lambda+p\alpha_i}.
\end{aligned}$$

Repeating in this way we get

$$\begin{aligned}
&T_\omega F_i^{(p)} \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong (T_{n-1}) \dots (T_{n-i} \dots T_{n-1}) F_{n-1}^{(p)} (T_{n-i-1} \dots T_{n-1}) \dots (T_1 \dots T_{n-1}) \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong (T_{n-i})(T_{n-i+1}T_{n-i}) \dots (T_{n-1} \dots T_{n-i}) F_{n-1}^{(p)} (T_{n-i-1} \dots T_{n-1}) \dots (T_1 \dots T_{n-1}) \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong (T_{n-i})(T_{n-i+1}T_{n-i}) \dots F_{n-2}^{(p)} (T_{n-1} \dots T_{n-i}) (T_{n-i-1} \dots T_{n-1}) \dots (T_1 \dots T_{n-1}) \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong \dots \\
&\cong (T_{n-i}) F_{n-i}^{(p)} (T_{n-i+1}T_{n-i}) \dots (T_{n-1} \dots T_{n-i}) (T_{n-i-1} \dots T_{n-1}) \dots (T_1 \dots T_{n-1}) \mathbf{1}_{\lambda+p\alpha_i} \\
&\cong (T_{n-i}) \mathbf{1}_\mu F_{n-i}^{(p)} (T_{n-i})^{-1} T_\omega \mathbf{1}_{\lambda+p\alpha_i}.
\end{aligned}$$

Now $\mu = s_{n-i}s_\omega \cdot (\lambda + p\alpha_i) - p\alpha_{n-i}$ and subsequently

$$\begin{aligned}
\langle \mu, \alpha_{n-i} \rangle &= \langle \lambda + p\alpha_i, s_\omega s_{n-i} \cdot \alpha_{n-i} \rangle - 2p \\
&= -\langle \lambda + p\alpha_i, s_\omega \cdot \alpha_{n-i} \rangle - 2p \\
&= \langle \lambda + p\alpha_i, \alpha_i \rangle - 2p \\
&= \langle \lambda, \alpha_i \rangle
\end{aligned}$$

where we use that $\langle \cdot, \cdot \rangle$ is invariant under the action of the Weyl group and that $s_\omega \cdot \alpha_{n-i} = -\alpha_i$ for any $i = 1, \dots, n-1$. Finally, by Corollary 4.5, we get that

$$T_{n-i} \mathbf{1}_\mu F_{n-i}^{(p)} \cong E_{n-i}^{(p)} T_{n-i} \mathbf{1}_{\mu+p\alpha_{n-i}} \langle -p(\langle \lambda, \alpha_i \rangle + p + 1) \rangle [p]$$

which completes the proof of the first claim.

The proof of the second claim follows similarly. \square

Corollary 5.4. *Suppose $\mathfrak{g} = \mathfrak{sl}_n$ and fix $p \geq 0$. Then we have*

$$\begin{aligned}
T_\omega^2 \mathbf{1}_\lambda F_i^{(p)} &\cong F_i^{(p)} T_\omega^2 \mathbf{1}_{\lambda+p\alpha_i} \langle -2p(\langle \lambda, \alpha_i \rangle + p + 1) \rangle [2p] & \text{if } \langle \lambda, \alpha_i \rangle \geq 0 \\
T_\omega^2 \mathbf{1}_\lambda E_i^{(p)} &\cong E_i^{(p)} T_\omega^2 \mathbf{1}_{\lambda-p\alpha_i} \langle -2p(-\langle \lambda, \alpha_i \rangle + p + 1) \rangle [2p] & \text{if } \langle \lambda, \alpha_i \rangle \leq 0.
\end{aligned}$$

Proof. We prove the first assertion (the second follows similarly). Recall that by Proposition 5.3 we have

$$T_\omega \mathbf{1}_\lambda F_i^{(p)} \cong \mathbf{1}_{s_\omega \cdot \lambda} E_{n-i}^{(p)} T_\omega \langle -p(\langle \lambda, \alpha_i \rangle + p + 1) \rangle [p].$$

Now

$$\langle s_\omega \cdot \lambda, \alpha_{n-i} \rangle = \langle \lambda, s_\omega \cdot \alpha_{n-i} \rangle = -\langle \lambda, \alpha_i \rangle$$

so we can apply Proposition 5.3 again to get

$$T_\omega \mathbf{1}_{s_\omega \cdot \lambda} E_{n-i}^{(p)} \cong \mathbf{1}_\lambda F_i^{(p)} T_\omega \langle -p(\langle \lambda, \alpha_i \rangle + p + 1) \rangle [p]$$

and the result follows. \square

Let us denote by $U_i \mathbf{1}_\mu = \mathbf{1}_{s_i \cdot \mu} U_i$ the map

$$E_i^{(-\langle \mu, \alpha_i \rangle)} \mathbf{1}_\mu \text{ if } \langle \mu, \alpha_i \rangle \leq 0 \quad \text{and} \quad F_i^{(\langle \mu, \alpha_i \rangle)} \mathbf{1}_\mu \text{ if } \langle \mu, \alpha_i \rangle \geq 0.$$

Corollary 5.5. *Suppose $\mathfrak{g} = \mathfrak{sl}_n$. Then we have $T_\omega U_i \mathbf{1}_\lambda \cong U_{n-i} T_\omega \mathbf{1}_\lambda$ and subsequently*

$$(33) \quad T_\omega^2 U_i \mathbf{1}_\lambda \cong U_i T_\omega^2 \mathbf{1}_\lambda.$$

Proof. Suppose $\langle \lambda, \alpha_i \rangle \geq 0$ so that $U_i \mathbf{1}_\lambda \cong F_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda$ (the case $\langle \lambda, \alpha_i \rangle \leq 0$ is the same). Then using the same argument as in the proof of Proposition 5.3 we find that

$$T_\omega U_i \mathbf{1}_\lambda \cong T_\omega F_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda \cong T_{n-i} F_{n-i}^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\mu T_{n-i}^{-1} T_\omega \cong T_{n-i} U_{n-i} \mathbf{1}_\mu T_{n-i}^{-1} T_\omega$$

where to get the last isomorphisms we use that $\mu = s_{n-i} s_\omega \cdot \lambda$ so that

$$\langle \mu, \alpha_{n-i} \rangle = \langle s_{n-i} s_\omega \cdot \lambda, \alpha_{n-i} \rangle = -\langle \lambda, s_\omega \cdot \alpha_{n-i} \rangle = \langle \lambda, \alpha_i \rangle.$$

By Corollary 4.5 we have $T_{n-i} U_{n-i} \mathbf{1}_\mu \cong U_{n-i} T_{n-i} \mathbf{1}_\mu$ so we get that

$$T_\omega U_i \mathbf{1}_\lambda \cong T_{n-i} U_{n-i} T_{n-i}^{-1} T_\omega \mathbf{1}_\lambda \cong U_{n-i} T_\omega \mathbf{1}_\lambda.$$

Applying this result twice gives (33). □

Now let us denote by $R\mathbf{1}_\lambda := \text{Cone}(\mathbf{1}_\lambda \rightarrow T_\omega^2 \mathbf{1}_\lambda)$.

Proposition 5.6. *If $\ell \geq 0$ then the complexes $T_\omega^{2\ell} R\mathbf{1}_\lambda$ and $RT_\omega^{2\ell} \mathbf{1}_\lambda$ are supported in homological degrees $\leq -2\ell$.*

Proof. We deal with the complex $T_\omega^{2\ell} R\mathbf{1}_\lambda$ (the proof for the other complex is the same). The idea is to use the expression for $\mathbf{1}_\lambda T_\omega^2$ from (32) to study $T_\omega^{2\ell} \mathbf{1}_\lambda T_\omega^2$. Consider the left most term $\mathbf{1}_\lambda T_{n-1}$ and suppose $\langle \lambda, \alpha_{n-1} \rangle \leq 0$. Then $\mathbf{1}_\lambda T_{n-1}$ is given by a complex

$$\cdots \rightarrow \mathbf{1}_\lambda E_{n-1}^{(s)} F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle + s)} \langle -s \rangle \rightarrow \mathbf{1}_\lambda E_{n-1}^{(s-1)} F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle + s-1)} \langle -s+1 \rangle \rightarrow \cdots \rightarrow \mathbf{1}_\lambda F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle)}.$$

Then by Corollary 5.4 we have

$$(34) \quad T_\omega^{2\ell} \mathbf{1}_\lambda E_{n-1}^{(s)} F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle + s)} \cong E_{n-1}^{(s)} T_\omega^{2\ell} \mathbf{1}_{\lambda - s\alpha_{n-1}} F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle + s)} \langle -2s\ell(-\langle \lambda, \alpha_{n-1} \rangle + s+1) \rangle [2s\ell]$$

which is a complex supported in homological degrees $\leq -2s\ell$. Thus only the term $T_\omega^{2\ell} \mathbf{1}_\lambda F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle)}$ can contribute to the cohomology in degrees $\geq -2\ell$. In this case, by Corollary 5.5, we have

$$(35) \quad T_\omega^{2\ell} \mathbf{1}_\lambda F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle)} \cong F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle)} T_\omega^{2\ell} \mathbf{1}_\lambda.$$

Now we repeat this argument with the other T s in the expression for $\mathbf{1}_\lambda T_\omega^2$ and conclude that the only terms in $\mathbf{1}_\lambda T_\omega^2$ which can contribute something in cohomology degrees $\geq -2\ell$ in $T_\omega^{2\ell} \mathbf{1}_\lambda T_\omega^2$ is

$$[(U_{n-1})(U_{n-2}U_{n-1}) \cdots (U_1 \cdots U_{n-1})][(U_{n-1} \cdots U_1) \cdots (U_{n-1}U_{n-2})(U_{n-1})] \mathbf{1}_\lambda T_\omega^{2\ell}.$$

Using Corollary 5.5 we can rewrite this as

$$(36) \quad [(U_{n-1})(U_{n-2}U_{n-1}) \cdots (U_1 \cdots U_{n-2})] T_\omega^{2\ell} U_{n-1}^2 [(U_{n-2} \cdots U_1) \cdots (U_{n-1}U_{n-2})(U_{n-1})] \mathbf{1}_\lambda.$$

Consider the middle factor $U_{n-1}^2 \mathbf{1}_\mu$ in (36) where $\mu = s_{n-1} s_\omega \cdot \lambda$. Let us suppose $\langle \mu, \alpha_{n-1} \rangle \geq 0$ (the case $\langle \mu, \alpha_{n-1} \rangle \leq 0$ is the same). Then

$$\begin{aligned} U_{n-1} U_{n-1} \mathbf{1}_\mu &\cong E_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle)} F_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle)} \mathbf{1}_\mu \\ &\cong \bigoplus_{j \geq 0} \bigoplus_{\langle \mu, \alpha_{n-1} \rangle - j} \mathbf{1}_\mu F_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle - j)} E_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle - j)} \mathbf{1}_\mu. \end{aligned}$$

Now, by the same argument as above (using Corollary 5.4),

$$T_\omega^{2\ell} \mathbf{1}_\mu F_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle - j)} E_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle - j)} \cong F_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle - j)} T_\omega^{2\ell} E_{n-1}^{(\langle \mu, \alpha_{n-1} \rangle - j)} \mathbf{1}_\mu \langle -2p\ell(\langle \mu, \alpha_{n-1} \rangle + p+1) \rangle [2p\ell]$$

where $p = \langle \mu, \alpha_{n-1} \rangle - j$. This is supported in cohomological degrees $\leq -2\ell$ unless $j = \langle \mu, \alpha_{n-1} \rangle$ which leaves us with one copy of the identity. Thus the only terms in (36) which could contribute to cohomological degrees $> -2\ell$ come from

$$[(U_{n-1})(U_{n-2}U_{n-1}) \dots (U_1 \dots U_{n-2})]T_\omega^{2\ell} \mathbf{1}_\mu [(U_{n-2} \dots U_1) \dots (U_{n-1}U_{n-2})(U_{n-1})] \mathbf{1}_\lambda.$$

Repeating this argument with U_{n-2}, U_{n-3} and so on, we are left with just one term, namely $T_\omega^{2\ell} \mathbf{1}_\lambda$. But this term in $T_\omega^{2\ell} R\mathbf{1}_\lambda = T_\omega^{2\ell} \text{Cone}(\mathbf{1}_\lambda \rightarrow T_\omega^2 \mathbf{1}_\lambda)$ is zero since it is exactly the one that comes from the term $\mathbf{1}_\lambda$ inside the complex $T_\omega^2 \mathbf{1}_\lambda$. Thus $T_\omega^{2\ell} R\mathbf{1}_\lambda$ is supported in homological degrees $\leq -2\ell$. \square

Proposition 5.6 implies that $\lim_{\ell \rightarrow \infty} T_\omega^{2\ell} R\mathbf{1}_\lambda = 0$. This means that $P^- := \lim_{\ell \rightarrow \infty} T_\omega^{2\ell}$ exists (the negative in P^- indicates that the complex is bounded above). Sometimes we will denote P^- by T_ω^∞ to remind us of its definition as an infinite twist.

Proposition 5.7. *If $\mathfrak{g} = \mathfrak{sl}_n$ then $P^- \mathbf{1}_\lambda \in \text{Kom}^-(\mathcal{K})$ is an idempotent, meaning that $P^- \cdot P^- \cong P^-$.*

Proof. Consider the map $\phi : T_\omega^\infty \mathbf{1}_\lambda \rightarrow T_\omega^\infty T_\omega^\infty \mathbf{1}_\lambda$ induced by $\mathbf{1}_\lambda \rightarrow T_\omega^\infty \mathbf{1}_\lambda$. The cone of this map is (by definition) $T_\omega^\infty R\mathbf{1}_\lambda$. Now for any $\ell \geq 0$ we have $T_\omega^\infty T_\omega^{2\ell} \mathbf{1}_\lambda \cong T_\omega^\infty \mathbf{1}_\lambda$. Thus $T_\omega^\infty R\mathbf{1}_\lambda \cong T_\omega^\infty T_\omega^{2\ell} R\mathbf{1}_\lambda$ which, by applying Proposition 5.6 to $T_\omega^{2\ell} R\mathbf{1}_\lambda$, is supported in homological degrees $\leq -2\ell$. Since ℓ can be chosen arbitrarily it follows that $T_\omega^\infty R\mathbf{1}_\lambda$ is contractible and hence ϕ is an isomorphism. \square

5.4. Convergence of P^- in K -theory. We now show that $P^- \in \text{Kom}_*^-(\mathcal{K})$. This implies that P^- converges in K -theory and gives a well defined element $\check{p}([P^-]) \in \hat{K}(\mathcal{K})$.

To do this we use the canonical basis \dot{B} of $\dot{U}_q(\mathfrak{g})$ defined in [Lu]. We are acting on the finite dimensional representation $\oplus_\lambda K(\mathcal{D}(\lambda))$ of $U_q(\mathfrak{g})$ which means that all but finitely many basis elements in \dot{B} act by zero (see [Lu] Remark 25.2.4 and Section 23.1.2). We denote the elements that act nontrivially by $\{b_1, \dots, b_l\} \subset \dot{B}$.

To simplify notation we represent P^- by some complex C and denote by $\langle C^{-u} \rangle_q$ the minimum power of q which occurs among all $f_i \in \mathbb{C}[q, q^{-1}]$ if we write $[C^{-u}] = \sum_i f_i b_i$ in the basis above. What we need to show is that

$$\lim_{u \rightarrow \infty} \langle C^{-u} \rangle_q = \infty.$$

Let L denote the minimum of

$$\langle A_1 \dots A_m b_j A'_1 \dots A'_{m'} \mathbf{1}_\lambda \rangle_q$$

where each A and A' is either $E_k^{(e_k)}$ or $F_k^{(e_k)}$ for some $k \in I$ and $e_k \in \mathbb{N}$. This is well defined since the set of such elements is finite (there are only a finite number of nonzero weights).

Now let us study $T_\omega^{2\ell} R\mathbf{1}_\lambda$ where, as before, $R\mathbf{1}_\lambda = \text{Cone}(\mathbf{1}_\lambda \rightarrow T_\omega^2 \mathbf{1}_\lambda)$. We proceed as in the proof of Proposition 5.6. Namely we first consider $T_\omega^{2\ell} \mathbf{1}_\lambda T_{n-1}$ which, assuming $\langle \lambda, \alpha_{n-1} \rangle \leq 0$, is made up of terms of the form $T_\omega^{2\ell} \mathbf{1}_\lambda E_{n-1}^{(s)} F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle + s)}$ (the case $\langle \lambda, \alpha_{n-1} \rangle \geq 0$ is similar). Moving the $E_{n-1}^{(s)}$ term over to the other side of $T_\omega^{2\ell}$ we get equation (34). Thus

$$T_\omega^{2\ell} \mathbf{1}_\lambda E_{n-1}^{(s)} F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle + s)} [T_{n-1}^{-1} T_\omega^2]$$

is homotopic to

$$(37) \quad E_{n-1}^{(s)} T_\omega^{2\ell} \mathbf{1}_{\lambda - s\alpha_{n-1}} F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle + s)} T_{n-1}^{-1} T_\omega^2 [-2s\ell(-\langle \lambda, \alpha_{n-1} \rangle + s + 1)][2s\ell].$$

Now suppose that we know by induction that $T_\omega^{2\ell}$ is homotopic to a complex $C_{2\ell}$ such that $\inf_u \langle C_{2\ell}^{-u} \rangle_q \geq m_\ell$ for some m_ℓ . Then the complex in (37) is isomorphic to a complex $C_{2\ell, s}$ with

$$\inf_u \langle C_{2\ell, s}^{-u} \rangle_q \geq 2s\ell(-\langle \lambda, \alpha_{n-1} \rangle + s + 1) + L + m_\ell \geq 2s\ell(s + 1) + L + m_\ell$$

for any u . The right side is at least $4\ell + L + m_\ell$ unless $s = 0$ in which case we get

$$F_{n-1}^{(-\langle \lambda, \alpha_{n-1} \rangle)} T_\omega^{2\ell} T_{n-1}^{-1} T_\omega^2 \cong U_{n-1} T_\omega^{2\ell} T_{n-2} T_{n-2}^{-1} T_{n-1}^{-1} T_\omega^2$$

and we repeat the argument above. In this way we eventually end up with

$$[(U_{n-1})(U_{n-2}U_{n-1}) \dots (U_1 \dots U_{n-1})][(U_{n-1} \dots U_1) \dots (U_{n-1}U_{n-2})(U_{n-1})]1_\lambda T_\omega^{2\ell}$$

which we deal with as in the proof of Proposition 5.6.

In conclusion, we find that $T_\omega^{2\ell} R1_\lambda$ is homotopic to a complex $\hat{C}_{2(\ell+1)}$ such that

$$(38) \quad \inf_u \langle \hat{C}_{2(\ell+1)}^{-u} \rangle_q \geq 4\ell + L + m_\ell.$$

In particular, this means that $T_\omega^{2(\ell+1)}$ is homotopic to a complex $C_{2(\ell+1)}$ such that

$$\inf_u \langle C_{2(\ell+1)}^{-u} \rangle_q \geq \min(4\ell + L + m_\ell, m_\ell).$$

The right side above is equal to m_ℓ if $\ell > -L/4$. In other words, for $\ell > -L/4$ we have $\inf_u \langle C_{2(\ell+1)}^{-u} \rangle_q \geq m$ for some fixed m . Subsequently (38) implies that

$$(39) \quad \inf_u \langle \hat{C}_{2(\ell+1)}^{-u} \rangle_q \geq 4\ell + L + m$$

if $\ell > -L/4$.

Now suppose $\lim_{u \rightarrow \infty} \langle C^{-u} \rangle_q \neq 0$. Then there exists some l and $\ell_1 \leq \ell_2 \leq \dots$ where $\ell_i \rightarrow \infty$ so that $\langle C_{2\ell'_i}^{-\ell_i} \rangle_q \leq l$ for some $\ell'_i \rightarrow \infty$. We can assume that ℓ'_i is the smallest integer with this property, meaning that $\langle C_{2(\ell'_i-1)}^{-\ell_i} \rangle_q > l$. Then from the exact triangle

$$C_{2(\ell'_i-1)} \rightarrow C_{2\ell'_i} \rightarrow \hat{C}_{2\ell'_i}$$

it follows that $\langle \hat{C}_{2\ell'_i}^{-\ell_i} \rangle_q \leq l$ which contradicts (39). Thus $P^- \in \text{Kom}_*^-(\mathcal{K})$.

This, together with the other results from this section, proves Theorem 2.2.

6. CATEGORIFIED CLASPS

In this section we prove Theorem 2.3.

Denote by \mathbb{C}^m the standard representation of $U_q(\mathfrak{sl}_m)$ with basis v_1, \dots, v_m . The vector space

$$\Lambda_q(\mathbb{C}^m) = \mathbb{C}[q, q^{-1}] \langle v_1, \dots, v_m \rangle / (v_i^2, v_i v_j + q v_j v_i \text{ for } i < j)$$

is the standard wedge product representation of $U_q(\mathfrak{sl}_m)$.

Now consider the $U_q(\mathfrak{sl}_m)$ -module $\Lambda_q^{\underline{i}}(\mathbb{C}^m) := \Lambda_q^{i_1}(\mathbb{C}^m) \otimes \dots \otimes \Lambda_q^{i_n}(\mathbb{C}^m)$ where \underline{i} is a sequence of integers $0 \leq i_1, \dots, i_n \leq m$. In terms of highest weight representations $\Lambda_q^{i_k}(\mathbb{C}^m) = V_{\Lambda_{i_k}}$. As a $U_q(\mathfrak{sl}_m)$ -module $\Lambda_q^{\underline{i}}(\mathbb{C}^m)$ has a unique summand isomorphic to the highest weight representation $V_{\underline{i}} := V_{\sum_k \Lambda_{i_k}}$. We denote by

$$\iota : V_{\underline{i}} \rightarrow \Lambda_q^{\underline{i}}(\mathbb{C}^m) \text{ and } \pi : \Lambda_q^{\underline{i}}(\mathbb{C}^m) \rightarrow V_{\underline{i}}$$

the natural inclusion and projection. Their composition is denoted

$$P1_{\underline{i}} := \iota \circ \pi \in \text{End}_{U_q(\mathfrak{sl}_m)}(\Lambda_q^{\underline{i}}(\mathbb{C}^m)).$$

Notice $\pi \circ \iota \in \text{End}_{U_q(\mathfrak{sl}_m)}(V_{\underline{i}})$ is a multiple of the identity since $V_{\underline{i}}$ is irreducible. So $(P1_{\underline{i}})^2$ is a multiple of $P1_{\underline{i}}$ and we uniquely rescale $P1_{\underline{i}}$ so that $(P1_{\underline{i}})^2 = P1_{\underline{i}}$. Following [Kup] we refer to the idempotent $P1_{\underline{i}}$ as a *clasp*.

6.1. Skew Howe duality. Our aim now is to understand clasps using skew Howe duality, *i.e.* by studying $\Lambda_q(\mathbb{C}^m \otimes \mathbb{C}^n)$. Formally, this algebra is the quadratic dual of the quantum algebra $M_q(m \times n)$ which is the quantum analogue of the algebra of $m \times n$ matrices (see [Man], in particular section 8.9). There exist two isomorphisms

$$\Lambda_q(\mathbb{C}^m)^{\otimes n} \longleftarrow \Lambda_q(\mathbb{C}^m \otimes \mathbb{C}^n) \longrightarrow \Lambda_q(\mathbb{C}^n)^{\otimes m}.$$

For our purposes, we only identify the composition isomorphism

$$(40) \quad \Lambda_q(\mathbb{C}^m)^{\otimes n} \longrightarrow \Lambda_q(\mathbb{C}^n)^{\otimes m}$$

(as $\mathbb{C}[q, q^{-1}]$ -modules) as follows. Having fixed the basis v_1, \dots, v_m for \mathbb{C}^m the left side of (40) has basis $\{v_{\underline{i}_1} \otimes \dots \otimes v_{\underline{i}_n}\}$ where each \underline{i}_k is a sequence $1 \leq s_1 < \dots < s_l \leq m$ and $v_{\underline{i}_k} = v_{s_1} \wedge \dots \wedge v_{s_l}$. Likewise, the right side of (40) has basis $\{w_{\underline{j}_1} \otimes \dots \otimes w_{\underline{j}_m}\}$ where w_1, \dots, w_n is a basis of \mathbb{C}^n . Then the map in (40) is given by

$$v_{\underline{i}_1} \otimes \dots \otimes v_{\underline{i}_n} \mapsto (-1)^{\#\{(a,b,k_1,k_2): a \in \underline{i}_{k_1}, b \in \underline{i}_{k_2}, a < b, k_1 < k_2\}} w_{\underline{j}_1} \otimes \dots \otimes w_{\underline{j}_m}$$

where $\ell \in \underline{j}_{\ell'}$ if and only if $\ell' \in \underline{i}_{\ell}$.

Lemma 6.1. *The actions of $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_n)$ on $\Lambda_q(\mathbb{C}^m)^{\otimes n} \cong \Lambda_q(\mathbb{C}^n)^{\otimes m}$ commute.*

Proof. It is enough to consider the root $U_q(\mathfrak{sl}_2)$ subalgebras of $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_n)$. So we can assume $m = n = 2$ and that case can be checked by an explicit calculation. \square

For $N \in \mathbb{N}$ denote by $\Lambda_q^N(\mathbb{C}^m)^{\otimes n}$ the N -graded piece of $\Lambda_q(\mathbb{C}^m)^{\otimes n}$ where $\deg(v_i) = 1$. The action of $U_q(\mathfrak{sl}_n)$ preserves this piece. The decomposition of $\Lambda_q^N(\mathbb{C}^m)^{\otimes n}$ into weight spaces is given by $\oplus_{\underline{i}} \Lambda_q^{\underline{i}}(\mathbb{C}^m)$ where the direct sum is over all $0 \leq i_1, \dots, i_n \leq m$ with $\sum_j i_k = N$.

Under this action we have

$$E_k : \Lambda_q^{\underline{i}}(\mathbb{C}^m) \rightarrow \Lambda_q^{\underline{i} + \alpha_k}(\mathbb{C}^m) \text{ and } F_k : \Lambda_q^{\underline{i} + \alpha_k}(\mathbb{C}^m) \rightarrow \Lambda_q^{\underline{i}}(\mathbb{C}^m)$$

where $\alpha_k = (0, \dots, 0, -1, 1, 0, \dots, 0)$ with the nonzero entries in the k and $k+1$ spot. The dominant weights correspond to those \underline{i} where $0 \leq i_1 \leq \dots \leq i_n \leq m$ (in this case we call \underline{i} dominant). The notation $\mathbf{1}_{\underline{i}}$ indicates the projection onto this weight space.

Proposition 6.2. *If \underline{i} is a dominant weight then $P\mathbf{1}_{\underline{i}}F_k = 0$ for $k = 1, \dots, m-1$. Moreover, $P\mathbf{1}_{\underline{i}}$ is the unique such (nonzero) projection in $\text{End}_{U_q(\mathfrak{sl}_m)}(\Lambda_q^{\underline{i}}(\mathbb{C}^m))$.*

Proof. Since $\Lambda_q^{\underline{i} + \alpha_k}(\mathbb{C}^m)$ does not contain $V_{\underline{i}}$ as a direct summand it follows that $P\mathbf{1}_{\underline{i}}F_k = 0$ for any $k = 1, \dots, m-1$.

Now, the $U_q(\mathfrak{sl}_m)$ -module $\Lambda_q^{\underline{i} + \alpha_k}(\mathbb{C}^m)$ breaks up as a direct sum $V_{\underline{i}} \oplus V'$ for some $U_q(\mathfrak{sl}_m)$ -module V' . On the other hand, it is spanned by vectors $F_k(v)$ and highest weight vectors. By Lemma 6.3 the highest weight vectors span precisely $V_{\underline{i}}$. This means that if $P'\mathbf{1}_{\underline{i}}$ is a projector such that $P'\mathbf{1}_{\underline{i}}F_k = 0$ for all k then it must either be zero or it must project onto $V_{\underline{i}}$ (in which case $P' = P$). \square

Lemma 6.3. *The $U_q(\mathfrak{sl}_m)$ -submodule $V_{\underline{i}} \subset \Lambda_q^{\underline{i}}(\mathbb{C}^m)$ coincides with the vector space of highest weight vectors for the action of $U_q(\mathfrak{sl}_n)$.*

Proof. In order to prove this it suffices to consider the case $q = 1$. Now, skew Howe duality says that, as an $(\mathfrak{sl}_m, \mathfrak{sl}_n)$ -bimodule, we have

$$\Lambda^N(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \oplus_{|\underline{i}|=N} V_{\underline{i}^\vee} \boxtimes V_{\underline{i}}$$

where the sum is over all partitions (or equivalently Young diagrams) \underline{i} of size N which fit in an $n \times m$ box and \underline{i}^\vee denotes the dual Young diagram (obtained by flipping about a diagonal). In particular, this means that the highest weight vectors of the isotypic component $V_{\underline{i}^\vee}$ for the action of \mathfrak{sl}_n is spanned by $V_{\underline{i}}$. \square

6.2. The action of T_ω . Consider again the action of $U_q(\mathfrak{sl}_n)$ on $\Lambda_q^N(\mathbb{C}^m \otimes \mathbb{C}^n)$ described above. The decategorification of (4) and (5) gives

$$(41) \quad T_k \mathbf{1}_{\underline{i}} := \sum_{s \geq 0} (-q)^s E_k^{(-\langle \underline{i}, \alpha_k \rangle + s)} F_k^{(s)} \mathbf{1}_{\underline{i}} \quad \text{or} \quad T_k \mathbf{1}_{\underline{i}} := \sum_{s \geq 0} (-q)^s F_k^{(\langle \underline{i}, \alpha_k \rangle + s)} E_k^{(s)} \mathbf{1}_{\underline{i}}$$

depending on whether $\langle \underline{i}, \alpha_k \rangle \leq 0$ or $\langle \underline{i}, \alpha_k \rangle \geq 0$ (recall that the shift $\langle 1 \rangle$ decategorifies to multiplication by q^{-1}). Here the pairing $\langle \cdot, \cdot \rangle$ is given by the usual dot product on $\underline{i} = (i_1, \dots, i_n)$ and $\alpha_k = (0, \dots, -1, 1, \dots, 0)$. Note that $T_k \mathbf{1}_{\underline{i}} = \mathbf{1}_{s_k \cdot \underline{i}} T_k$ where s_k acts on \underline{i} by permuting the k and $(k+1)$ st entries. We define T_ω , just like T_ω as

$$T_\omega \mathbf{1}_{\underline{i}} := (T_{n-1} \dots T_1)(T_{n-1} \dots T_2) \dots (T_{n-1} T_{n-2})(T_{n-1}) \mathbf{1}_{\underline{i}}.$$

Remark 6.4. This expression for $T_k \mathbf{1}_{\underline{i}}$ differs from the one in section 5.2.1 of [Lu] which defines it as

$$\sum_{a, b, c \geq 0, a-b+c=\langle \underline{i}, \alpha_k \rangle} (-1)^b q^{ac-b} E_k^{(a)} F_k^{(b)} E_k^{(c)} \mathbf{1}_{\underline{i}}.$$

Equation (41) is (modulo some extra q factors) a sum over the terms above when $c = 0$. One can actually try to categorify the other terms in the summation but one gets a complex homotopic to zero. This is not obvious and will be discussed elsewhere. So in essence, the categorifications of the summation above and of (41) are the same.

Proposition 6.5. *Suppose \underline{i} is dominant and let v be a highest weight vector such that $F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v)$ has weight \underline{i} . Then*

$$(42) \quad T_\omega F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v) = q^{\langle \underline{i}' + \underline{i}, \underline{i}' - \underline{i} \rangle + 2 \sum_l p_l} F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v)$$

where \underline{i}' is the weight of v .

Proof. The decategorification of Corollary 5.4 states that

$$T_\omega^2 \mathbf{1}_{\underline{i}} F_k^{(p)} \cong q^{2p(\langle \underline{i}, \alpha_k \rangle + p + 1)} F_k^{(p)} T_\omega^2 \mathbf{1}_{\underline{i}}.$$

Applying this repeatedly and keeping track of the powers of q we find that the exponent of q is

$$2p_1(\langle \underline{i}, \alpha_{k_1} \rangle + p_1 + 1) + 2p_2(\langle \underline{i} + p_1 \alpha_{k_1}, \alpha_{k_2} \rangle + p_2 + 1) + \dots + 2p_j(\langle \underline{i} + p_1 \alpha_{k_1} + \dots + p_{j-1} \alpha_{k_{j-1}}, \alpha_{k_j} \rangle + p_j + 1)$$

which simplifies to

$$2 \sum_l p_l \langle \underline{i}, \alpha_{k_l} \rangle + 2 \sum_{1 \leq a < b \leq j} \langle p_a \alpha_{k_a}, p_b \alpha_{k_b} \rangle + 2 \sum_l (p_l^2 + p_l).$$

Now using that $\langle \alpha_k, \alpha_k \rangle = 2$ and $\underline{i}' - \underline{i} = \sum_l p_l \alpha_{k_l}$ it is easy to check that this simplifies to give

$$2 \langle \underline{i}, \underline{i}' - \underline{i} \rangle + \langle \underline{i}' - \underline{i}, \underline{i}' - \underline{i} \rangle + 2 \sum_l p_l$$

which is the same as the exponent of q in (42).

Finally, to complete the proof one shows that $T_\omega^2 v = v$ since v is a highest weight vector. To see this first note that if $E_k(w) = 0$ (resp. $F_k(w) = 0$) then $T_k(w) = U_k(w)$ and $F_k U_k(w) = 0$ (resp. $E_k U_k(w) = 0$). Here U_k is the decategorification of U_k , namely $U_k(w)$ equals

$$E_k^{(-\langle \mathbf{wt}(w), \alpha_k \rangle)}(w) \text{ if } \langle \mathbf{wt}(w), \alpha_k \rangle \leq 0 \quad \text{and} \quad F_k^{(\langle \mathbf{wt}(w), \alpha_k \rangle)}(w) \text{ if } \langle \mathbf{wt}(w), \alpha_k \rangle \geq 0$$

where $\mathbf{wt}(w)$ denotes the weight of w . Thus we get:

$$T_\omega^2(v) = [(U_{n-1})(U_{n-2} U_{n-1}) \dots (U_1 \dots U_{n-1})][(U_{n-1} \dots U_1) \dots (U_{n-1} U_{n-2})(U_{n-1})](v).$$

Now consider the middle two terms $U_{n-1}U_{n-1}(w)$ where $w = (U_{n-2} \dots U_1) \dots (U_{n-1}U_{n-2})(U_{n-1})(v)$. By the argument above we know either $E_{n-1}(w) = 0$ or $F_{n-1}(w) = 0$. Let us suppose $E_{n-1}(w) = 0$ (the other case is the same). Then

$$\begin{aligned} U_{n-1}U_{n-1}(w) &= E_{n-1}^{(\langle \text{wt}(w), \alpha_{n-1} \rangle)} F_{n-1}^{(\langle \text{wt}(w), \alpha_{n-1} \rangle)}(w) \\ &= \sum_{j \geq 0} \left[\begin{matrix} \langle \text{wt}(w), \alpha_{n-1} \rangle \\ j \end{matrix} \right] F_{n-1}^{(\langle \text{wt}(w), \alpha_{n-1} \rangle - j)} E_{n-1}^{(\langle \text{wt}(w), \alpha_{n-1} \rangle - j)}(w) \end{aligned}$$

Since $E_{n-1}(w) = 0$ all these terms vanish except for the one term when $j = \langle \text{wt}(w), \alpha_{n-1} \rangle$. Thus we get $U_{n-1}U_{n-1}(w) = w$. Repeating this way we obtain $T_\omega^2(v) = v$. \square

Corollary 6.6. *Suppose \underline{i} is a dominant weight. The clasp $P\mathbf{1}_{\underline{i}}$ is the unique (nonzero) projection in $\text{End}(\Lambda_q^{\underline{i}}(\mathbb{C}^m))$ which satisfies $T_\omega^2 P = P = PT_\omega^2$.*

Proof. The vector space $\Lambda_q^{\underline{i}}(\mathbb{C}^m)$ is spanned by vectors $v \in V_{\underline{i}} \subset \Lambda_q^{\underline{i}}(\mathbb{C}^m)$ (which are highest weight vectors) and vectors of the form $F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v)$ where v is a highest weight vector.

In the first case we have $P(v) = v$ and $T_\omega^2(v) = v$ by Proposition 6.5. In the second case we have $PF_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v) = 0$ by Proposition 6.2 and likewise $PT_\omega^2 F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v) = 0$ by Proposition 6.5. This shows that $T_\omega^2 P = P = PT_\omega^2$.

On the other hand, suppose P' is another projection which satisfies $T_\omega^2 P' = P' = P'T_\omega^2$. Then

$$\begin{aligned} P' F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v) &= P' T_\omega^2 F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v) \\ &= P' q^{\langle \underline{i}' + \underline{i}, \underline{i}' - \underline{i} \rangle + 2 \sum_l p_l} F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v) \end{aligned}$$

by Proposition 6.5. Now $\langle \underline{i}', \underline{i}' - \underline{i} \rangle \geq 0$ and $\langle \underline{i}, \underline{i}' - \underline{i} \rangle \geq 0$ since $\underline{i}, \underline{i}'$ are dominant weights and $\underline{i}' - \underline{i} = \sum_l p_l \alpha_{i_l}$ where $p_l > 0$. Thus the exponent of q is positive and we conclude $P' F_{k_1}^{(p_1)} \dots F_{k_j}^{(p_j)}(v) = 0$. Then $P'\mathbf{1}_{\underline{i}} F_k = 0$ for any k and hence $P'\mathbf{1}_{\underline{i}} = P\mathbf{1}_{\underline{i}}$ by Proposition 6.2. \square

Recall that $P^{-1}\mathbf{1}_{\underline{i}} \in \text{Kom}^-(\mathcal{K})$ is an idempotent (Proposition 5.7) satisfying $T_\omega^2 P^{-1}\mathbf{1}_{\underline{i}} \cong P^{-1}\mathbf{1}_{\underline{i}} \cong T_\omega^2 P^{-1}\mathbf{1}_{\underline{i}}$ (essentially by definition). It follows that if \underline{i} is a dominant weight then $P^{-1}\mathbf{1}_{\underline{i}}$ categorifies $P\mathbf{1}_{\underline{i}}$ in the sense that $\tilde{p}([P^{-1}\mathbf{1}_{\underline{i}}]) = P\mathbf{1}_{\underline{i}} \in \hat{K}(\mathcal{K})$.

If \underline{i} is not dominant let $\sigma \in S_n$ be a permutation such that $\sigma \cdot \underline{i}$ is dominant and consider its lift T_σ to the braid group. Now $T_\sigma^{-1} P\mathbf{1}_{\underline{i}} T_\sigma$ is idempotent and T_σ, T_σ^{-1} commute with T_ω^2 which means

$$T_\omega^2 (T_\sigma^{-1} P\mathbf{1}_{\underline{i}} T_\sigma) \mathbf{1}_{\sigma \cdot \underline{i}} \cong T_\sigma^{-1} P\mathbf{1}_{\underline{i}} T_\sigma \mathbf{1}_{\sigma \cdot \underline{i}} \cong (T_\sigma^{-1} P\mathbf{1}_{\underline{i}} T_\sigma) T_\omega^2 \mathbf{1}_{\sigma \cdot \underline{i}}.$$

So by Corollary 6.6 we conclude that $T_\sigma^{-1} P\mathbf{1}_{\underline{i}} T_\sigma = P\mathbf{1}_{\sigma \cdot \underline{i}}$.

A similar argument show that $T_\sigma^{-1} P^{-1}\mathbf{1}_{\underline{i}} T_\sigma$ is idempotent and

$$T_\omega^2 (T_\sigma^{-1} P^{-1}\mathbf{1}_{\underline{i}} T_\sigma) \mathbf{1}_{\sigma \cdot \underline{i}} \cong T_\sigma^{-1} P^{-1}\mathbf{1}_{\underline{i}} T_\sigma \mathbf{1}_{\sigma \cdot \underline{i}} \cong (T_\sigma^{-1} P^{-1}\mathbf{1}_{\underline{i}} T_\sigma) T_\omega^2 \mathbf{1}_{\sigma \cdot \underline{i}}.$$

This means that $\tilde{p}([T_\sigma^{-1} P^{-1}\mathbf{1}_{\underline{i}} T_\sigma]) = P\mathbf{1}_{\sigma \cdot \underline{i}}$ which, since $T_\sigma^{-1} P\mathbf{1}_{\underline{i}} T_\sigma = P\mathbf{1}_{\sigma \cdot \underline{i}}$, implies $\tilde{p}([P^{-1}\mathbf{1}_{\underline{i}}]) = P\mathbf{1}_{\underline{i}}$. This concludes the proof of Theorem 2.3.

7. THE REPRESENTATION $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ AND TANGLE INVARIANTS

We will now prove Theorem 2.4. Most of the work is setting everything up correctly at the decategorified level (subsections 7.2 and 7.3). It is then straightforward to pass to categories (subsection 7.4). In subsection 7.5 we explain how to obtain \mathbb{Z}^2 -graded homological link invariants from our setup.

7.1. The weight spaces. If we fix m, N then

$$\begin{aligned}\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N}) &\cong \Lambda_q^{mN}(\mathbb{C}^m \oplus \dots \oplus \mathbb{C}^m) \\ &\cong \bigoplus_{i_1 + \dots + i_{2N} = mN} \Lambda_q^{i_1}(\mathbb{C}^m) \otimes \dots \otimes \Lambda_q^{i_{2N}}(\mathbb{C}^m)\end{aligned}$$

where there are $2N$ summands \mathbb{C}^m in the first line. As noted in section 6.1 each $\Lambda_q^{i_1}(\mathbb{C}^m) \otimes \dots \otimes \Lambda_q^{i_{2N}}(\mathbb{C}^m)$ is a weight space for the action of $U_q(\mathfrak{sl}_{2N})$ where

$$\dots \otimes \Lambda_q^{i_k}(\mathbb{C}^m) \otimes \Lambda_q^{i_{k+1}}(\mathbb{C}^m) \otimes \dots \xrightleftharpoons[F_k]{E_k} \dots \otimes \Lambda_q^{i_k-1}(\mathbb{C}^m) \otimes \Lambda_q^{i_{k+1}+1}(\mathbb{C}^m) \otimes \dots$$

Thus, we get:

Lemma 7.1. *The nonzero weight spaces of $\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$ as a $U_q(\mathfrak{sl}_{2N})$ -module are in natural bijection with $2N$ -tuples (i_1, \dots, i_{2N}) where $0 \leq i_1, \dots, i_{2N} \leq m$ and $\sum_{\ell} i_{\ell} = mN$. The generators E_k and F_k of \mathfrak{sl}_{2N} correspond to maps*

$$(i_1, \dots, i_k, i_{k+1}, \dots, i_{2N}) \xrightleftharpoons[F_k]{E_k} (i_1, \dots, i_k - 1, i_{k+1} + 1, \dots, i_{2N}).$$

while the Weyl group of \mathfrak{sl}_{2N} , with generators s_1, \dots, s_{2N-1} , permutes the weights:

$$s_k \cdot (i_1, \dots, i_k, i_{k+1}, \dots, i_{2N}) = (i_1, \dots, i_{k+1}, i_k, \dots, i_{2N}).$$

The weight space of $\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$ labeled by $\underline{i} = (i_1, \dots, i_{2N})$ will be denoted $V(\underline{i})$. In this notation $\alpha_k = (0, \dots, -1, 1, \dots, 0)$ while the bilinear form $\langle \cdot, \cdot \rangle$ on weights is given by the usual dot product $\langle \underline{i}, \underline{i}' \rangle = \underline{i} \cdot \underline{i}'$ of tuples. The highest weight is $(0, \dots, 0, m, \dots, m)$ where there are N 0s and ms .

Given a weight \underline{i} denote by $\rho(\underline{i})$ the sequence obtained by dropping all $i_{\ell} \in \{0, m\}$. For example, if $\underline{i} = (1, 0, 3, m, 5)$ then $\rho(\underline{i}) = (1, 3, 5)$. Moreover, we denote by $S(\underline{i})$ the set of weights \underline{i}' such that $\rho(\underline{i}) = \rho(\underline{i}')$. We can act on $S(\underline{i})$ by exchanging $i_{\ell}, i_{\ell+1} \in \underline{i}$ if at least one of them belongs to $\{0, m\}$. This action is clearly transitive.

Lemma 7.2. *Fix \underline{i} . The action described above on weights lifts via the braid group action to an action on $\bigoplus_{\underline{j} \in S(\underline{i})} V(\underline{j})$ which canonically identifies all the weight spaces $V(\underline{j})$ with $\underline{j} \in S(\underline{i})$.*

Proof. Recall that the braid group action on weight spaces is induced by $T_k \mathbf{1}_{\underline{i}} : V(\underline{i}) \rightarrow V(s_k \cdot \underline{i})$ where

$$T_k \mathbf{1}_{\underline{i}} := \sum_{s \geq 0} (-q)^s E_k^{(-\langle \underline{i}, \alpha_k \rangle + s)} F_k^{(s)} \mathbf{1}_{\underline{i}} \quad \text{or} \quad T_k \mathbf{1}_{\underline{i}} := \sum_{s \geq 0} (-q)^s F_k^{(\langle \underline{i}, \alpha_k \rangle + s)} E_k^{(s)} \mathbf{1}_{\underline{i}}$$

depending on whether $\langle \underline{i}, \alpha_k \rangle \leq 0$ or $\langle \underline{i}, \alpha_k \rangle \geq 0$. Now, suppose that i_k or i_{k+1} belongs to $\{0, m\}$. Then it is easy to see that all the terms in the summation are zero except when $s = 0$. In other words, $T_k \mathbf{1}_{\underline{i}}$ equals $E_k^{(-\langle \underline{i}, \alpha_k \rangle)} \mathbf{1}_{\underline{i}}$ or $F_k^{(\langle \underline{i}, \alpha_k \rangle)} \mathbf{1}_{\underline{i}}$ depending on whether $\langle \underline{i}, \alpha_k \rangle \leq 0$ or $\langle \underline{i}, \alpha_k \rangle \geq 0$.

This means that $T_k^2 \mathbf{1}_{\underline{i}}$ is either

$$F_k^{(-\langle \underline{i}, \alpha_k \rangle)} E_k^{(-\langle \underline{i}, \alpha_k \rangle)} \mathbf{1}_{\underline{i}} \text{ or } E_k^{(\langle \underline{i}, \alpha_k \rangle)} F_k^{(\langle \underline{i}, \alpha_k \rangle)} \mathbf{1}_{\underline{i}}.$$

In the second case

$$E_k^{(\langle \underline{i}, \alpha_k \rangle)} F_k^{(\langle \underline{i}, \alpha_k \rangle)} \mathbf{1}_{\underline{i}} = \sum_{j \geq 0} \begin{bmatrix} \langle \underline{i}, \alpha_k \rangle \\ j \end{bmatrix} F_k^{(\langle \underline{i}, \alpha_k \rangle - j)} E_k^{(\langle \underline{i}, \alpha_k \rangle - j)} \mathbf{1}_{\underline{i}} = \mathbf{1}_{\underline{i}}$$

since $E_k^{(\langle \underline{i}, \alpha_k \rangle - j)} \mathbf{1}_{\underline{i}} = 0$ for $j < \langle \underline{i}, \alpha_k \rangle$ (and likewise in the first case). So $T_k^2 \mathbf{1}_{\underline{i}} = \mathbf{1}_{\underline{i}}$.

Since the action is transitive on $S(\underline{i})$ we can assume that

$$\underline{i} = (0, \dots, 0, m, \dots, m, i_1, \dots, i_j)$$

where there are n_1 0's and n_2 m 's and $i_\ell \notin \{0, m\}$. Now $T_k \mathbf{1}_{\underline{i}}$ clearly acts by the identity if $k = 1, \dots, n_1 - 1$ or $k = n_1 + 1, \dots, n_1 + n_2 - 1$ (*i.e.* if it acts just on the 0's or just on the m 's). Combining this with the fact that $T_k^2 \mathbf{1}_{\underline{i}} = \mathbf{1}_{\underline{i}}$ if i_k or i_{k+1} is in $\{0, m\}$ (proven above) gives the result. \square

Now let $N \rightarrow \infty$. We denote the resulting (infinite dimensional) vector space by $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$, but this is just notation. Its definition is by imitating the weight space decomposition above. More precisely,

$$\begin{aligned} \Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty}) &\cong \Lambda_q^{m\infty}\left(\bigoplus_{k \in \mathbb{Z}} \mathbb{C}^m\right) \\ &\cong \bigoplus_{\underline{i}} [\dots \otimes \Lambda_q^{i_k}(\mathbb{C}^m) \otimes \Lambda_q^{i_{k+1}}(\mathbb{C}^m) \otimes \dots] \end{aligned}$$

where the direct sum is over all sequences \underline{i} where $i_k = 0$ if $k \ll 0$ and $i_k = m$ if $k \gg 0$ and the sum of all $i_k \notin \{0, m\}$ is divisible by m . Because of the former condition the infinite tensor product in each summand above is actually finite. As before, E_k and F_k correspond to maps

$$(\dots, i_k, i_{k+1}, \dots) \xrightleftharpoons[F_k]{E_k} (\dots, i_k - 1, i_{k+1} + 1, \dots).$$

except now $k \in \mathbb{Z}$. The weight space labeled by \underline{i} is still denoted $V(\underline{i})$.

Given \underline{i} we again have $\rho(\underline{i})$ which forgets all the terms in \underline{i} equal to 0 or m . Note that $\rho(\underline{i})$ is still a finite sequence. As before, we denote by $S(\underline{i})$ all sequences \underline{i}' such that $\rho(\underline{i}) = \rho(\underline{i}')$.

Now consider the embedding $U_q(\mathfrak{sl}_{2N}) \rightarrow U_q(\mathfrak{sl}_\infty)$ given by $E_k \mapsto E_{k-N}$ and $F_k \mapsto F_{k-N}$ where $k = 1, \dots, 2N - 1$. If we restrict $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ to $U_q(\mathfrak{sl}_{2N})$ we find that it contains the module $\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$ as a direct summand. Moreover, given any weight \underline{i} , the weight space $V(\underline{i})$ sits as the weight space of such a direct summand for N sufficiently large. The following is an immediate corollary of Lemma 7.2

Corollary 7.3. *Using the braid group action, any two weight spaces $V(\underline{i}_1)$ and $V(\underline{i}_2)$ of $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ are canonically isomorphic if $\underline{i}_1, \underline{i}_2 \in S(\underline{i})$.*

7.2. Tangle invariants: fundamental representations. Let us briefly recall what we mean by an \mathfrak{sl}_m framed tangle invariant. Consider a framed tangle T whose strands are labeled by representations (or equivalently, dominant weights) of \mathfrak{sl}_m . Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ be the labels on the strands at the top and $\underline{\mu} = (\mu_1, \dots, \mu_{n'})$ be the labels on the strands at the bottom. As usual, V_λ denotes the irreducible \mathfrak{sl}_m representation with highest weight λ .

A $U_q(\mathfrak{sl}_m)$ framed tangle invariant associates to such a tangle a map of $U_q(\mathfrak{sl}_m)$ -modules

$$\psi(T) : V_{\underline{\lambda}} := V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \rightarrow V_{\underline{\mu}} := V_{\mu_1} \otimes \dots \otimes V_{\mu_{n'}}.$$

This is done by analyzing a tangle projection from bottom to top and assigning maps to each cap, cup and crossing (the generating tangles are shown in figure 1). If the maps $\psi(T)$ do not depend on the planar projection of the tangle then we get a map

$$\psi : \left\{ (\underline{\lambda}, \underline{\mu}) \text{ tangles} \right\} \rightarrow \text{Hom}_{U_q(\mathfrak{sl}_m)}(V_{\underline{\lambda}}, V_{\underline{\mu}})$$

If $T = K$ is a framed, oriented link then $\psi(K)$ is a map $\mathbb{C}[q, q^{-1}] \rightarrow \mathbb{C}[q, q^{-1}]$ and $\psi(K)(1)$ becomes a polynomial invariant of K .

For the moment we examine the case when the λ_i are all fundamental weights. In this case, $\underline{\lambda} = (\Lambda_{i_1}, \dots, \Lambda_{i_n})$ and hence $V_{\underline{\lambda}}$ is identified with $V(\underline{i})$ where $\underline{i} = (\dots, 0, i_1, \dots, i_n, m, \dots)$.

Reshetikhin and Turaev defined such a framed tangle invariant in [RT]. We now explain how to recover their maps from the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ using skew Howe duality. First we define the following:

- two maps $V(\underline{i}) \rightarrow V(s_k \cdot \underline{i})$ corresponding to the two crossings in figure (1)
- two maps $V(\dots, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots) \rightleftharpoons V(\dots, i_{k-1}, i_{k+2}, \dots)$ where $i_k + i_{k+1} = m$ corresponding to the cap and cup in figure (1).

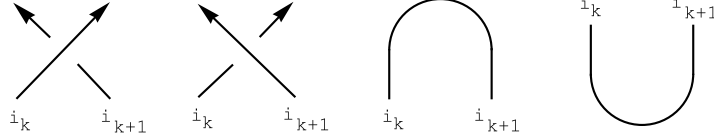


FIGURE 1. The cup and cap can have either orientation as long as $i_k + i_{k+1} = m$.

The first two maps are defined by $T_k \mathbf{1}_{\underline{i}} : V(\underline{i}) \rightarrow V(s_k \cdot \underline{i})$ and its inverse $T_k^{-1} \mathbf{1}_{\underline{i}}$. The cap map is given by

$$E_k^{(i_k)} \mathbf{1}_{\underline{i}} : V(\dots, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots) \rightarrow V(\dots, i_{k-1}, 0, m, i_{k+2}, \dots)$$

where we then identify $V(\dots, i_{k-1}, 0, m, i_{k+2}, \dots)$ with $V(\dots, i_{k-1}, i_{k+2}, \dots)$ using Corollary 7.3 (alternatively, we could have defined the cap as $F_k^{(i_{k+1})} \mathbf{1}_{\underline{i}}$ which yields the same map). Similarly, the cup map is defined by

$$\mathbf{1}_{\underline{i}} F_k^{(i_k)} : V(\dots, i_{k-1}, 0, m, i_{k+2}, \dots) \rightarrow V(\dots, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots)$$

where we identify $V(\dots, i_{k-1}, i_{k+2}, \dots)$ with $V(\dots, i_{k-1}, 0, m, i_{k+2}, \dots)$.

Proposition 7.4. *These maps define a $U_q(\mathfrak{sl}_m)$ framed tangle invariant.*

Proof. In Lemma 6.1 we saw that the actions of $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_n)$ on $\Lambda_q^N(\mathbb{C}^m \otimes \mathbb{C}^n)$ commute. This implies that the actions of $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_\infty)$ on $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ also commute. Since $V(\underline{i})$ is a weight space of $U_q(\mathfrak{sl}_\infty)$ the action of $U_q(\mathfrak{sl}_m)$ preserves it. Moreover, all the maps defined above (crossings, caps and cups) are written in terms of elements in $U_q(\mathfrak{sl}_\infty)$ which means that they commute with the $U_q(\mathfrak{sl}_m)$ action and hence induce maps of $U_q(\mathfrak{sl}_m)$ -modules.

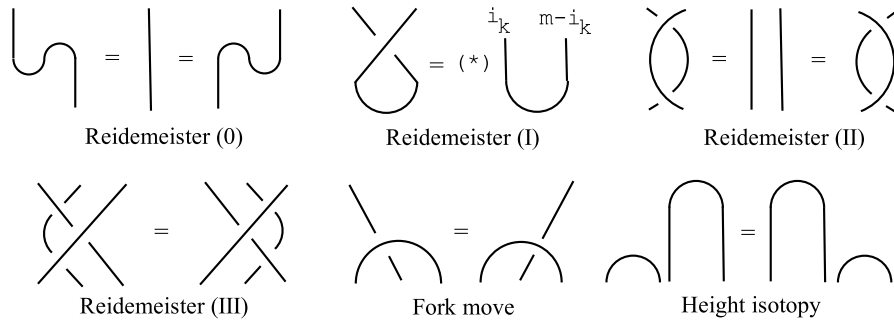


FIGURE 2. Invariance relations where $(\star) = q^{i_k(m-i_k)}(-q)^{\min(i_k, m-i_k)}$.

To show that we have an invariant it suffices to check the relations in figure (2) where the strands are labeled by arbitrary fundamental weights. One also needs to check isotopy relations involving changing

the height of crossings, caps and cups which are far apart (for example, the right most relation in the second line of figure (2)). However, these isotopy relations are clear because the functors E_i, F_i commute with E_j, F_j if $|i - j| > 1$. The other relations above are equivalent to the following identities.

- **(R0)**: The compositions

$$\begin{aligned} V(\dots, 0, m, i_k, \dots) &\xrightarrow{F_{k-2}^{(i_k)}} V(\dots, i_k, m - i_k, i_k, \dots) \xrightarrow{F_{k-1}^{(i_k)}} V(\dots, i_k, m, 0, \dots) \\ V(\dots, i_k, m, 0, \dots) &\xrightarrow{E_{k-1}^{(i_k)}} V(\dots, i_k, m - i_k, i_k, \dots) \xrightarrow{E_{k-2}^{(i_k)}} V(\dots, 0, m, i_k, \dots) \end{aligned}$$

are both equal to the identity map if you identify $V(\dots, 0, m, i_k, \dots)$ and $V(\dots, i_k, m, 0, \dots)$.

- **(RI)**: The composition

$$V(\dots, m, 0, \dots) \xrightarrow{E_k^{(i_k)}} V(\dots, m - i_k, i_k, \dots) \xrightarrow{T_k} V(\dots, i_k, m - i_k, \dots)$$

is equal to $(\star) = q^{i_k(m-i_k)}(-q)^{\min(i_k, m-i_k)}$ times the cup map

$$E_k^{(m-i_k)} : V(\dots, m, 0, \dots) \rightarrow V(\dots, i_k, m - i_k, 0, \dots).$$

- **(RII)**: The maps $V(\underline{i}) \xrightarrow{T_k} V(s_k \cdot \underline{i})$ and $V(s_k \cdot \underline{i}) \xrightarrow{T_k^{-1}} V(\underline{i})$ are inverses of each other.
- **(RIII)**: The compositions

$$\begin{aligned} V(\underline{i}) &\xrightarrow{T_k} V(s_k \cdot \underline{i}) \xrightarrow{T_{k+1}} V(s_{k+1}s_k \cdot \underline{i}) \xrightarrow{T_k} V(s_k s_{k+1} s_k \cdot \underline{i}) \\ V(\underline{i}) &\xrightarrow{T_{k+1}} V(s_{k+1} \cdot \underline{i}) \xrightarrow{T_k} V(s_k s_{k+1} \cdot \underline{i}) \xrightarrow{T_{k+1}} V(s_{k+1} s_k s_{k+1} \cdot \underline{i}) \end{aligned}$$

are equal to each other.

- **Fork move**: The compositions

$$\begin{aligned} V(\dots, i_k, i_{k+1}, m - i_k, \dots) &\xrightarrow{T_k} V(\dots, i_{k+1}, i_k, m - i_k, \dots) \xrightarrow{E_{k+1}^{(i_k)}} V(\dots, i_{k+1}, 0, m, \dots) \\ V(\dots, i_k, i_{k+1}, m - i_k, \dots) &\xrightarrow{T_{k+1}^{-1}} V(\dots, i_k, m - i_k, i_{k+1}, \dots) \xrightarrow{E_k^{(i_k)}} V(\dots, 0, m, i_{k+1}, \dots) \end{aligned}$$

are equal to each other if you identify $V(\dots, i_{k+1}, 0, m, \dots)$ and $V(\dots, 0, m, i_{k+1}, \dots)$.

(R0). Consider the first composition. The identification

$$V(\dots, 0, m, i_k, \dots) \longleftrightarrow V(\dots, i_k, m, 0, \dots)$$

can be done by the element $T_{k-1}T_{k-2}T_{k-1}$. Thus we need to show that

$$T_{k-1}T_{k-2}T_{k-1}F_{k-1}^{(i_k)}F_{k-2}^{(i_k)} = \text{id} \in \text{End}_{U_q(\mathfrak{sl}_m)}(V(\dots, 0, m, i_k, \dots)).$$

Fortunately, in this case every T has only one term in its summation and we get

$$\begin{aligned} T_{k-1}T_{k-2}T_{k-1}F_{k-1}^{(i_k)}F_{k-2}^{(i_k)} &= F_{k-1}^{(m-i_k)}E_{k-2}^{(i_k)}E_{k-1}^{(m)}F_{k-1}^{(i_k)}F_{k-2}^{(i_k)} \\ &= F_{k-1}^{(m-i_k)}E_{k-2}^{(i_k)}E_{k-1}^{(m-i_k)}F_{k-2}^{(i_k)} \\ &= F_{k-1}^{(m-i_k)}E_{k-2}^{(i_k)}F_{k-2}^{(i_k)}E_{k-1}^{(m-i_k)} \\ &= F_{k-1}^{(m-i_k)}E_{k-1}^{(m-i_k)} \\ &= \text{id} \end{aligned}$$

where we use Lemma 4.1 (and careful tracking of the weights) to get equalities two, four and five and that $E_i^{(a)}F_j^{(b)} = F_j^{(b)}E_i^{(a)}$ if $i \neq j$ to get the third equality. Similarly one can prove that the second composition of (R0) is equal to the identity.

(RI). Suppose $2i_k \leq m$. Then

$$\begin{aligned} T_k E_k^{(i_k)} \mathbf{1}_{\underline{i}} &= F_k^{(i_k)} T_k q^{i_k(m-i_k+1)} (-1)^{i_k} \mathbf{1}_{\underline{i}} \\ &= F_k^{(i_k)} E_k^{(m)} q^{i_k(m-i_k)} (-q)^{i_k} \mathbf{1}_{\underline{i}} \\ &= E_k^{(m-i_k)} q^{i_k(m-i_k)} (-q)^{i_k} \mathbf{1}_{\underline{i}} \end{aligned}$$

where $\underline{i} = (\dots, m, 0, \dots)$. To get the first equality we use Corollary 4.5 where the shift $\langle -i_k(m-i_k+1) \rangle [i_k]$ is translated in K-theory using $(1) \leftrightarrow q^{-1}$. The third equality is by Lemma 4.1.

On the other hand, if $2i_k \geq m$ then we have

$$\begin{aligned} T_k E_k^{(i_k)} \mathbf{1}_{\underline{i}} &= T_k F_k^{(m-i_k)} E_k^{(m)} \mathbf{1}_{\underline{i}} \\ &= E_k^{(m-i_k)} T_k E_k^{(m)} q^{(m-i_k)(i_k+1)} (-1)^{m-i_k} \mathbf{1}_{\underline{i}} \\ &= E_k^{(m-i_k)} q^{i_k(m-i_k)} (-q)^{(m-i_k)} \mathbf{1}_{\underline{i}}. \end{aligned}$$

(RII) and (RIII). The (RII) relation follows from [CKL3] where we show that T_k is invertible. The (RIII) relation follows from [CaK3] (see the discussion in section 5.1 and Proposition 5.1).

Fork move. The identification

$$V(\dots, i_{k+1}, 0, m, \dots) \longleftrightarrow V(\dots, 0, m, i_{k+1}, \dots)$$

can be done by the element $T_k T_{k+1}$. Thus we need to show that

$$E_{k+1}^{(i_k)} T_k \mathbf{1}_{\underline{i}} = T_k T_{k+1} E_k^{(i_k)} T_{k+1}^{-1} \mathbf{1}_{\underline{i}}$$

where $\underline{i} = (\dots, i_k, i_{k+1}, m-i_k, \dots)$. By (the decategorification of) Lemma 5.2, $T_k T_{k+1} E_k^{(i_k)} = E_{k+1}^{(i_k)} T_k T_{k+1}$ and the result follows. \square

Above we used the $U_q(\mathfrak{sl}_\infty)$ structure to construct an isomorphism $V_{\underline{\lambda}} = V(\underline{i}) \rightarrow V(s_k \cdot \underline{i}) = V_{\underline{\mu}}$ and to define caps and cups. Alternatively, one can define this isomorphism as $\text{Flip} \circ R$ where R is the R -matrix defined in [RT] and Flip exchanges factors k and $k+1$ (one can also define caps and cups in this way). We now explain why these two approaches are equivalent.

Proposition 7.5. *The maps $T_k \mathbf{1}_{\underline{i}}$, $E_k^{(i_k)} \mathbf{1}_{\underline{i}}$ and $\mathbf{1}_{\underline{i}} F_k^{(i_k)}$ used above to define crossings, cups and caps are equal to the Reshetikhin-Turaev maps [RT] up to a scalar and multiplication by some power of q .*

Proof. By [CKL1, Thm. 4.3] we know that the crossings are assigned the same map up to an (explicitly identified) multiple of q and a sign. Now, the cap (and similarly the cup) is defined in both cases by maps $\Lambda_q^i(\mathbb{C}^m) \otimes \Lambda_q^{m-i}(\mathbb{C}^m) \rightarrow \mathbb{C}$ of $U_q(\mathfrak{sl}_m)$ -modules. If $q = 1$ the space of such maps is one-dimensional. Hence the map above and that in [RT] must differ only by some factor $f \in \mathbb{C}[q, q^{-1}]$ (everything is defined over $\mathbb{C}[q, q^{-1}]$). Because of Reidemeister move (0) it follows that f must be invertible which means that f is some power of q (up to a scalar). \square

Remark 7.6. Once you fix the maps associated to $(i, m-i)$ crossings there is not much choice for the maps associated to an $(i, m-i)$ cup or cap if you insist that the invariant of the unknot labeled by i or $m-i$ equals $\begin{bmatrix} m \\ i \end{bmatrix}$. More precisely, the possible choices are parametrized by a scalar and a power of q (i.e. an integer). This is because, once you rescale an $(i, m-i)$ cup by some $f \in \mathbb{C}[q, q^{-1}]$ (which is the only way to change it since the space of maps is one-dimensional) then using Reidemeister (0) we find that all other $(i, m-i)$ cups must also be scaled by f and all $(m-i, i)$ caps are scaled by f^{-1} . Moreover, the evaluation of the unknot implies that all other $(i, m-i)$ caps also get scaled by f^{-1} and hence all $(m-i, i)$ cups are scaled by f .

Since any link contains an equal number of caps and cups this means that the link invariant is unchanged by such a rescaling. In particular, the link invariant obtained using the maps $T_k, E_k^{(i_k)}$ and $F_k^{(i_k)}$ above only differs from the Reshetikhin-Turaev link invariant by (at worse) a scalar and some power of q .

7.3. Tangle invariants: arbitrary representations. To obtain the Reshetikhin-Turaev invariants for tangles labeled by arbitrary weights we use the clasps $P\mathbf{1}_{\underline{i}}$ defined in section 6. Recall that $P\mathbf{1}_{\underline{i}}$ is the composition

$$(43) \quad \Lambda_q^{\underline{i}}(\mathbb{C}^m) \xrightarrow{\pi} V_{\underline{i}} \xrightarrow{\iota} \Lambda_q^{\underline{i}}(\mathbb{C}^m)$$

where $V_{\underline{i}} = V_{\sum_k \Lambda_{i_k}}$, the first map is projection and the second map is inclusion. Such a map is unique up to scalar but if we insist that $P^2 = P$ then this scalar is also uniquely determined.

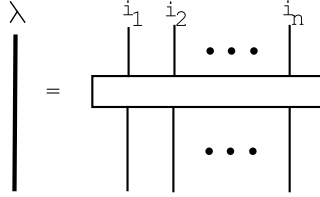


FIGURE 3. A strand labeled $\lambda = \sum_{k=1}^n \Lambda_{i_k}$ defined using a clasp.

Instead of dealing with a strand labeled by $\lambda = \sum_{k=1}^n \Lambda_{i_k}$ we replace it with n strands labeled by i_1, \dots, i_n together with a clasp. Diagrammatically, the clasp is illustrated by a box as in figure (3). In this setup, we work with clasps and strands labeled only by fundamental weights. One can then associate a map to crossings, cups and caps involving strands labeled by arbitrary representations as follows.

Since $P\mathbf{1}_{\underline{i}}$ is idempotent we recover $V_{\underline{i}}$ as the image of $P\mathbf{1}_{\underline{i}}$ on $\Lambda_q^{\underline{i}}(\mathbb{C}^m)$. Now, if $\sum_{k=1}^n \Lambda_{i_k}$ and $\mu = \sum_{k=1}^{n'} \Lambda_{j_k}$ one recovers the crossing map $V_{\underline{i}} \otimes V_{\underline{j}} \rightarrow V_{\underline{j}} \otimes V_{\underline{i}}$ from the crossing map $V(\underline{i}) \otimes V(\underline{j}) \rightarrow V(\underline{j}) \otimes V(\underline{i})$. Note that this latter map involves multiple crossings of strands labeled by fundamental representations and hence is somewhat complicated.

However, this construction only makes sense if this crossing map maps $V_{\underline{i}} \otimes V_{\underline{j}} \subset V(\underline{i}) \otimes V(\underline{j})$ to $V_{\underline{j}} \otimes V_{\underline{i}} \subset V(\underline{j}) \otimes V(\underline{i})$. This is indeed the case because of the relation in the first row of figure (4). The relations in figure (4) also ensure that the resulting maps satisfy the relations in figure (2) where $(\star) = q^{\sum_k i_k(m-i_k)}(-q)^{\sum_k \min(i_k, m-i_k)}$ if the top left strand is labeled by $\sum_k \Lambda_{i_k}$.

Proposition 7.7. *The relations in figure (4) are valid.*

Proof. This is a consequence of Proposition 7.10 where we check these relations at the categorical level. This means that we regard $P\mathbf{1}_{\underline{i}}$ as the class $[P^-\mathbf{1}_{\underline{i}}]$ where $P^- = \lim_{\ell \rightarrow \infty} T_{\omega}^{2\ell}$ rather than as the composition $\iota \circ \pi$ from (43). From this point of view the relations in figure (4) follow very easily. \square

7.4. Categorical knot invariants. Having constructed from the $U_q(\mathfrak{sl}_{\infty})$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ the Reshetikhin-Turaev framed tangle invariants we can now categorify the whole picture as follows. In place of $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ we take a categorical 2-representation whose weight spaces are graded categories $\mathcal{D}(\underline{i})$ such that $K(\mathcal{D}(\underline{i})) \cong V(\underline{i})$. In other words, suppose we have categorified the $U_q(\mathfrak{sl}_{\infty})$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$.

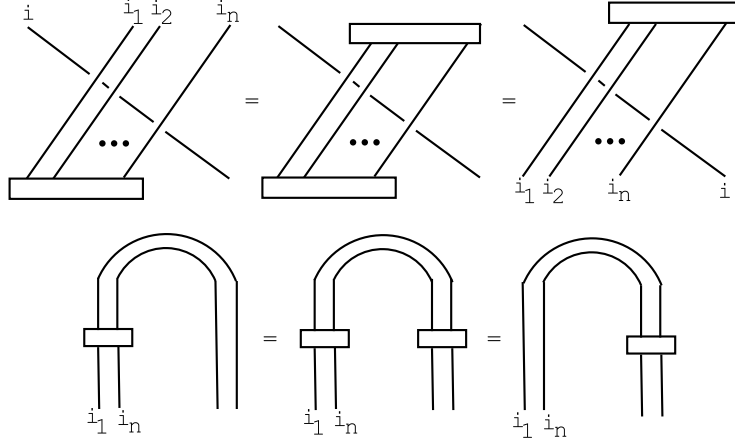


FIGURE 4. Relations involving clasps.

The isomorphisms $T_k : V(\underline{i}) \xrightarrow{\sim} V(s_k \cdot \underline{i})$ are categorified by complexes $\mathsf{T}_k \mathbf{1}_{\underline{i}}$ as in (4) and (5). Thus we actually need to use the homotopy categories $\mathsf{Kom}_*^-(\mathcal{D}(\lambda))$ instead of $\mathcal{D}(\lambda)$. So, to summarize, instead of weight spaces $V(\underline{i})$ and maps E_k, F_k we end up with:

$$\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_k, i_{k+1}, \dots)) \xrightleftharpoons[F_k]{E_k} \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_k - 1, i_{k+1} + 1, \dots))$$

and with $\mathsf{T}_k \mathbf{1}_{\underline{i}} : \mathsf{Kom}_*^-(\mathcal{D}(\underline{i})) \rightarrow \mathsf{Kom}_*^-(\mathcal{D}(s_k \cdot \underline{i}))$.

Corollary 7.8. *Any two categories $\mathsf{Kom}_*^-(\mathcal{D}(\underline{i}_1))$ and $\mathsf{Kom}_*^-(\mathcal{D}(\underline{i}_2))$ are canonically equivalent if $\underline{i}_1, \underline{i}_2 \in S(\underline{i})$.*

Proof. This is the categorical analogue of Corollary 7.3. The proof is the same. \square

Next we define functors associated to crossings, cups and caps as follows. More precisely, we need

- a functor $\mathsf{Kom}_*^-(\mathcal{D}(\underline{i})) \rightarrow \mathsf{Kom}_*^-(\mathcal{D}(s_k \cdot \underline{i}))$ which is invertible
- functors $\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots)) \rightleftarrows \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_{k+2}, \dots))$ whenever $i_k + i_{k+1} = m$.

Imitating what we did before, the first functor is $\mathsf{T}_k \mathbf{1}_{\underline{i}}$ which we know is invertible. The cap map is defined by

$$E_k^{(i_k)} \mathbf{1}_{\underline{i}} : \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots)) \rightarrow \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, 0, m, i_{k+2}, \dots))$$

where we then identify $\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, 0, m, i_{k+2}, \dots))$ with $\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_{k+2}, \dots))$ using Corollary 7.8 and the cup map is defined by

$$\mathbf{1}_{\underline{i}} F_k^{(i_k)} : \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, 0, m, i_{k+2}, \dots)) \rightarrow \mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots))$$

where we identify $\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, i_{k+2}, \dots))$ with $\mathsf{Kom}_*^-(\mathcal{D}(\dots, i_{k-1}, 0, m, i_{k+2}, \dots))$.

Proposition 7.9. *These maps define a homological framed tangle invariant. In other words, the relations in figure (2) hold as isomorphisms of functors, where*

$$(\star) = \langle -i_k(m - i_k) - \min(i_k, m - i_k) \rangle [\min(i_k, m - i_k)].$$

Proof. This is the categorical analogue of Proposition 7.4. The proof is the same. Recall that $\langle 1 \rangle$ denotes a grading shift by one (coming from the internal grading of the categories $\mathcal{D}(\lambda)$) while $[1]$ is a cohomological shift by one. The (RI) relation states that the curl can be undone up to twisting by (\star) , which is the categorical analogue of multiplying by $q^{i_k(m-i_k)}(-q)^{\min(i_k, m-i_k)}$. \square

Next we define the clasp using the complex of 1-morphisms $P^- \mathbf{1}_{\underline{i}} \in \text{Kom}_*^-(\mathcal{K})$. We already know that $P^- \mathbf{1}_{\underline{i}} P^- \mathbf{1}_{\underline{i}} \cong P^- \mathbf{1}_{\underline{i}}$ by Proposition 5.7. It remains to show that:

Proposition 7.10. *The relations in figure (4) are valid as isomorphisms of functors.*

Proof. We check the relation between the first two diagrams in the first row of figure (4) (the other cases are proved in exactly the same way). The left and right hand sides are equal to

$$(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)} \text{ and } (\mathsf{T}_\omega^\infty)(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)}$$

Now consider the map

$$(44) \quad (\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)} \longrightarrow (\mathsf{T}_\omega^\infty)(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)}$$

induced by the map $\mathbf{1} \rightarrow \mathsf{T}_\omega^\infty$. The cone of this is (by definition) the composition

$$(45) \quad (\mathsf{R})(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)}.$$

Now, for any $n \geq 0$

$$(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)} \cong (\mathsf{T}_\omega^{2n})(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)}$$

and $(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^{2n}) \cong (\mathsf{T}_\omega^{2n})(\mathsf{T}_1 \dots \mathsf{T}_n)$ so we get that

$$(\mathsf{R})(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)} \cong (\mathsf{R})(\mathsf{T}_\omega^{2n})(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty) \mathbf{1}_{(\underline{i}, i)}.$$

Now $(\mathsf{R})(\mathsf{T}_\omega^{2n})$ is supported in homological degrees $\leq -2n$ by Proposition 5.6. Since $(\mathsf{T}_1 \dots \mathsf{T}_n)(\mathsf{T}_\omega^\infty)$ is supported in non-positive degrees this means that (45) is supported in degrees $\leq -2n$. Since n above can be chosen arbitrarily this means that (45) is contractible and hence the map in (44) is an isomorphism. \square

Thus we get a categorical (framed) tangle invariant which categorifies the Reshetikhin-Turaev invariants. This concludes the proof of Theorem 2.4.

7.5. A homological link invariant. Using the process above, the homological invariant associated to a framed, oriented link K is a complex of functors

$$\Psi_-(K) \in \text{Kom}^-(\mathcal{D}(\underline{0}, \underline{m})) \rightarrow \text{Kom}^-(\mathcal{D}(\underline{0}, \underline{m}))$$

whose terms are direct sums of the identity functor with various grading shifts.

One way to obtain a doubly graded homology from $\Psi_-(K)$ is as follows. First, consider the \mathbb{Z} -graded algebra

$$A := \text{End}^*(\mathbf{1}_{(\underline{0}, \underline{m})}) = \bigoplus_{k \in \mathbb{Z}} \text{End}(\mathbf{1}_{(\underline{0}, \underline{m})}, \mathbf{1}_{(\underline{0}, \underline{m})} \langle k \rangle).$$

Now, an element of $\text{End}(\text{Kom}^-(\mathcal{D}(\underline{0}, \underline{m})))$ is a complex of functors

$$\dots \rightarrow A_{i-1} \rightarrow A_i \rightarrow A_{i+1} \rightarrow \dots \quad \text{where } A_i \in \text{End}(\mathcal{D}(\underline{0}, \underline{m})).$$

Thus we can consider the functor

$$\text{Hom}(\bigoplus_{k \in \mathbb{Z}} \mathbf{1}_{(\underline{0}, \underline{m})} \langle k \rangle, \bullet) : \text{End}(\text{Kom}^-(\mathcal{D}(\underline{0}, \underline{m}))) \rightarrow \text{Kom}^-(A\text{-mod}).$$

Since under this map $\mathbf{1}_{(\underline{0}, \underline{m})} \mapsto A$, the image $\Psi'_-(K)$ of $\Psi_-(K)$ is a complex of projective A -modules. Tensoring with \mathbb{C} gives us $\overline{\Psi}_-(K) := \mathbb{C} \otimes_A \Psi'_-(K)$ which is a complex of \mathbb{Z} -graded vector spaces. We can then define $\mathcal{H}_-^{i,j}(K)$ to be $\text{gr}^j H^i(\overline{\Psi}_-(K))$.

In this paper we will consider only 2-categories \mathcal{K} where $\text{End}^k(\mathbf{1}_{(\underline{0}, m)}) = 0$ if $k \neq 0$. Hence $A \cong \mathbb{C}$ and the procedure above is redundant (*i.e.* in this case $\Psi_-(K)$ contains the same information as $\mathcal{H}_-^{i,j}(K)$).

8. AFFINE GRASSMANNIANS AND THE 2-CATEGORY $\mathcal{K}_{\text{Gr}, m}$

We now define a categorical 2-representation $\mathcal{K}_{\text{Gr}, m}$ categorifying the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$. The 2-category $\mathcal{K}_{\text{Gr}, m}$ is obtained by considering derived categories of coherent sheaves on certain iterated Grassmannian bundles associated to the affine Grassmannian for PGL_m .

8.1. Derived categories of coherent sheaves. If X is a variety then we write $D(X)$ for the bounded derived category of coherent sheaves on X . The homological shift in $D(X)$ is denoted by $[-]$ and the Grothendieck group by $K(D(X))$ or $K(X)$ for short.

An object $\mathcal{P} \in D(X \times Y)$ whose support is proper over Y induces a Fourier-Mukai (FM) functor $\Phi_{\mathcal{P}} : D(X) \rightarrow D(Y)$ via $(\cdot) \mapsto \pi_{2*}(\pi_1^*(\cdot) \otimes \mathcal{P})$ (where every operation is derived). One says that \mathcal{P} is the (FM) kernel which induces $\Phi_{\mathcal{P}}$. The right and left adjoints $\Phi_{\mathcal{P}}^R$ and $\Phi_{\mathcal{P}}^L$ are induced by $\mathcal{P}_R := \mathcal{P}^\vee \otimes \pi_2^* \omega_X[\dim(X)]$ and $\mathcal{P}_L := \mathcal{P}^\vee \otimes \pi_1^* \omega_Y[\dim(Y)]$ respectively.

If $\mathcal{Q} \in D(Y \times Z)$ then $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{Q} * \mathcal{P}} : D(X) \rightarrow D(Z)$ where $\mathcal{Q} * \mathcal{P} = \pi_{13*}(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{Q})$ is the convolution product. So instead of talking about functors and compositions we will speak of kernels and convolutions.

If X carries a \mathbb{C}^\times action then we can consider the bounded derived category of \mathbb{C}^\times -equivariant coherent sheaves on X which, abusing notation, we also denote by $D(X)$. The sheaf $\mathcal{O}_X\{i\}$ denotes the structure sheaf of X shifted with respect to the \mathbb{C}^\times action so that if $f \in \mathcal{O}_X(U)$ is a local function then viewed as a section $f' \in \mathcal{O}_X\{i\}(U)$ we have $t \cdot f' = t^{-i}(t \cdot f)$. We denote by $\{i\}$ the operation of tensoring with $\mathcal{O}_X\{i\}$.

One can similarly define everything above for the bounded above (resp. below) derived categories $D^-(X)$ (resp. $D^+(X)$) of coherent sheaves on X .

8.2. Geometric categorical \mathfrak{g} actions. In [CaK3, Section 2.2.2] we introduced the idea of a geometric categorical \mathfrak{g} action, for any simply laced Kac-Moody algebra \mathfrak{g} . This definition gives rise to a categorical 2-representation of \mathfrak{g} as we now explain.

A geometric categorical \mathfrak{g} action consists of a sequence of varieties $Y(\lambda)$, FM kernels $\mathcal{E}_i^{(r)}, \mathcal{F}_i^{(r)}$ and deformations $\tilde{Y}(\lambda) \rightarrow \mathfrak{h}'$ of $Y(\lambda)$ where $\mathfrak{h}' := \text{span}\{\Lambda_i\}$ (everything is also equipped with a compatible \mathbb{C}^\times action). From these data we obtain categories $\mathcal{D}(\lambda) := D(Y(\lambda))$ and functors $\mathbf{E}_i^{(r)} := \Phi_{\mathcal{E}_i^{(r)}}, \mathbf{F}_i^{(r)} := \Phi_{\mathcal{F}_i^{(r)}}$.

Now, denote by $\tilde{Y}_i(\lambda) \rightarrow \mathbb{A}^1$ the restriction of $\tilde{Y}(\lambda)$ to the subspace in \mathfrak{h}' spanned by Λ_i . One can use $\tilde{Y}_i(\lambda)$ to obtain a morphism

$$\theta_i : \Delta_* \mathcal{O}_{Y(\lambda)} \rightarrow \Delta_* \mathcal{O}_{Y(\lambda)}[2]\{-2\} \in D(Y(\lambda) \times Y(\lambda))$$

where Δ denotes the diagonal embedding. This morphism is defined as the connecting map in the standard exact triangle

$$\Delta_* \mathcal{O}_{Y(\lambda)}[1]\{-2\} \rightarrow i^* i_* \Delta_* \mathcal{O}_{Y(\lambda)} \rightarrow \Delta_* \mathcal{O}_{Y(\lambda)}$$

where $i : Y(\lambda) \times Y(\lambda) \rightarrow \tilde{Y}_i(\lambda) \times Y(\lambda)$ is the natural inclusion of the first factor (see [CKL2, Sec. 5.1]). Since $\Delta_* \mathcal{O}_{Y(\lambda)}$ induces the identity functor on $D(Y(\lambda))$ we get a 2-morphism $\theta_i : \mathbf{1}_\lambda \rightarrow \mathbf{1}_\lambda\langle 2 \rangle$ where $\langle 1 \rangle := [1]\{-1\}$. Subsequently, the deformation $\tilde{Y}(\lambda)$ carries the same type of information as θ_i .

It is then not difficult to show that the relations in a categorical 2-representation are implied by those in a geometric categorical action.

8.3. A geometric categorical \mathfrak{sl}_{2N} action. Fix m . We will now define a geometric categorical \mathfrak{sl}_{2N} action which categorifies the $U_q(\mathfrak{sl}_{2N})$ -module $\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$. Recall that the weight spaces in this case are indexed by $\underline{i} = (i_1, \dots, i_{2N})$ where $0 \leq i_k \leq m$ and $\sum_k i_k = Nm$.

8.3.1. Varieties. We define

$$Y(\underline{i}) := \{\mathbb{C}[z]^m = L_0 \subset L_1 \subset \dots \subset L_{2N} \subset \mathbb{C}(z)^m : zL_j \subset L_{j-1}, \dim(L_j/L_{j-1}) = i_j\}$$

where the L_i are complex vector subspaces. If we forget L_{2N} then we get a Grassmannian bundle

$$Y(i_1, \dots, i_{2N-1}, i_{2N}) \rightarrow Y(i_1, \dots, i_{2N-1})$$

with fibres $\mathbb{G}(i_{2N}, m)$ because L_{2N} can be any subspace in $z^{-1}(L_{2N-1})/L_{2N-1} \cong \mathbb{C}^m$ of dimension i_{2N} . Thus $Y(\underline{i})$ is just an iterated Grassmannian bundle.

The relation of $Y(\underline{i})$ to the affine Grassmannian is via the twisted (or convolution) product. The variety $Y(i) \cong \mathbb{G}(i, m)$ is a subvariety of the affine Grassmannian for PGL_m . More generally, see Proposition 2.1 of [CaK2], $Y(\underline{i})$ is the twisted product

$$Y(\underline{i}) = Y(i_1) \tilde{\times} Y(i_2) \tilde{\times} \dots \tilde{\times} Y(i_{2N}).$$

For $j = 1, \dots, 2N$ there is a natural vector bundle on $Y(\underline{i})$ whose fibre over a point $\{L_0 \subset \dots \subset L_{2N}\}$ is the vector space L_j/L_0 . We will denote this bundle by \mathcal{L}_j .

8.3.2. \mathbb{C}^\times action. There is an action of \mathbb{C}^\times on $\mathbb{C}(z)$ given by $t \cdot z^k = t^{2k} z^k$. This induces an action of \mathbb{C}^\times on $\mathbb{C}(z)^m$. Since for any $v \in \mathbb{C}(z)^m$ we have $t \cdot (zv) = t^2 z(t \cdot v)$ this induces a \mathbb{C}^\times action on $Y(\underline{i})$ via $t \cdot (L_0, \dots, L_{2N}) = (t \cdot L_0, \dots, t \cdot L_{2N})$. Everything we define or claim will be \mathbb{C}^\times -equivariant with respect to this action. For example, the vector bundles \mathcal{L}_i defined above are \mathbb{C}^\times -equivariant, as are the correspondences in the next section.

8.3.3. Kernels. For $r \geq 0$ we define correspondences $W_k^r(\underline{i}) \subset Y(\underline{i}) \times Y(\underline{i} + r\alpha_k)$ as the subvariety $\{(L_\bullet, L'_\bullet) : L_\bullet \subset L'_\bullet\}$. Here, as in section 7.1, α_k denotes $(0, \dots, -1, 1, \dots, 0)$ where the -1 is the k th term. More explicitly,

$$\begin{aligned} W_k^r(\underline{i}) = \{ & \mathbb{C}[z]^m = L_0 \xrightarrow{i_1} \dots \xrightarrow{i_{k-1}} L_k \xrightarrow{r} L'_k \xrightarrow{i_{k+1}-r} L_{k+1} \xrightarrow{i_{k+1}} \dots \xrightarrow{i_{2N}} L_{2N} \subset \mathbb{C}(z)^m : \\ & zL_j \subset L_{j-1} \text{ for all } j, \text{ and } zL'_k \subset L_{k-1} \} \end{aligned}$$

where the arrows are inclusions and the superscripts indicate the codimension of the inclusion. We define the FM kernels

$$(46) \quad \mathcal{E}_k^{(r)} \mathbf{1}_{\underline{i}} := \mathcal{O}_{W_k^r(\underline{i})} \otimes \det(\mathcal{L}_{k+1}/\mathcal{L}'_k)^{-r} \otimes \det(\mathcal{L}_k/\mathcal{L}_{k-1})^r \{r(i_k - r)\} \in D(Y(\underline{i}) \times Y(\underline{i} + r\alpha_k))$$

$$(47) \quad \mathbf{1}_{\underline{i}} \mathcal{F}_k^{(r)} := \mathcal{O}_{W_k^r(\underline{i})} \otimes \det(\mathcal{L}'_k/\mathcal{L}_k)^{i_{k+1}-i_k+r} \{ri_{k+1}\} \in D(Y(\underline{i} + r\alpha_k) \times Y(\underline{i}))$$

where the prime denotes pullback from the second factor.

8.3.4. Deformations. The variety $Y(\underline{i})$ has a natural deformation over \mathbb{A}^{2N} given by

$$\{\mathbb{C}[z]^m = L_0 \subset L_1 \subset \dots \subset L_{2N} \subset \mathbb{C}(z)^m; \underline{x} \in \mathbb{A}^{2N} : (z - x_j)L_j \subset L_{j-1}, \dim(L_j/L_{j-1}) = i_j\}.$$

Notice that if $x_j = 0$ for all j we recover $Y(\underline{i})$. This deformation is trivial over the main diagonal in \mathbb{A}^{2N} . For this reason we restrict it to the locus where $x_{2N} = 0$ to obtain a deformation $\tilde{Y}(\underline{i}) \rightarrow \mathbb{A}^{2N-1}$. We identify \mathbb{A}^{2N-1} with $\mathfrak{h}' = X \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{2N}/(1, \dots, 1)$.

The \mathbb{C}^\times action on $Y(\underline{i})$ extends to a \mathbb{C}^\times action on all of $\tilde{Y}(\underline{i})$ if we act on the base \mathbb{A}^{2N-1} via $\underline{x} \mapsto t^2 \underline{x}$.

Theorem 8.1. *The data above gives a geometric categorical \mathfrak{sl}_{2N} action.*

Proof. All the \mathfrak{sl}_2 -type relations (conditions (i) to (vii) in [CaK3, Sec. 2]) were proven in [CKL1] as part of Theorem 3.3. Conditions (viii) to (xi) of [CaK3, Sec. 2] were checked in the case of cotangent bundles to partial flag varieties in [CaK3, Sec. 3]. The exact same proof applies here so we will only sketch it.

The first part of condition (viii) in [CaK3] consists of showing that

$$(48) \quad \mathcal{E}_k * \mathcal{E}_{k+1} * \mathcal{E}_k \mathbf{1}_{\underline{i}} \cong \mathcal{E}_k^{(2)} * \mathcal{E}_{k+1} \mathbf{1}_{\underline{i}} \oplus \mathcal{E}_{k+1} * \mathcal{E}_k^{(2)} \mathbf{1}_{\underline{i}}.$$

The right hand side can be computed explicitly. Namely $\mathcal{E}_k^{(2)} * \mathcal{E}_{k+1} \mathbf{1}_{\underline{i}}$ and $\mathcal{E}_{k+1} * \mathcal{E}_k^{(2)} \mathbf{1}_{\underline{i}}$ are equal to

$$\begin{aligned} & \mathcal{O}_{W_{k(2)k+1}}(\underline{i}) \otimes \det(\mathcal{L}_{k+2}/\mathcal{L}_{k+1})^{-1} \det(\mathcal{L}'_{k+1}/\mathcal{L}_k)^{-1} \det(\mathcal{L}'_k/\mathcal{L}_{k-1})^2 \{2i_k + i_{k+1} - 5\} \\ & \mathcal{O}_{W_{k+1k(2)}}(\underline{i}) \otimes \det(\mathcal{L}_{k+2}/\mathcal{L}_{i+1})^{-1} \det(\mathcal{L}_{k+1}/\mathcal{L}_k)^{-2} \det(\mathcal{L}'_{k+1}/\mathcal{L}'_k) \det(\mathcal{L}'_k/\mathcal{L}_{k-1})^2 \{2i_k + i_{k+1} - 3\} \end{aligned}$$

where

$$\begin{aligned} W_{k(2)k+1}(\underline{i}) &:= \{(L_\bullet, L'_\bullet) : L'_k \xrightarrow{2} L_k \xrightarrow{i_{k+1}-1} L'_{k+1} \xrightarrow{1} L_{k+1}, \text{ and } L_j = L'_j \text{ for } j \neq k, k+1\} \\ W_{k+1k(2)}(\underline{i}) &:= \{(L_\bullet, L'_\bullet) : L'_k \xrightarrow{2} L_k, L'_{k+1} \xrightarrow{1} L_{k+1}, zL_{k+1} \subset L'_k, \text{ and } L_j = L'_j \text{ for } j \neq k, k+1\} \end{aligned}$$

inside $Y(\underline{i}) \times Y(\underline{i} + 2\alpha_k + \alpha_{k+1})$. This is easy since, for instance, the intersection $\pi_{12}^{-1}(W_{k+1}) \cap \pi_{23}^{-1}(W_k^2)$ is of the expected dimension and the push-forward via π_{13} is one-to-one onto its image.

Similarly, and without any more difficulty, one can show that

$$(\mathcal{E}_k * \mathcal{E}_{k+1}) \mathbf{1}_{\underline{i} + \alpha_k} \cong \mathcal{O}_{W_{kk+1}} \otimes \det(\mathcal{L}_{k+2}/\mathcal{L}_{k+1})^\vee \det(\mathcal{L}'_k/\mathcal{L}_{k-1}) \{i_k + i_{k+1} - 2\}$$

where

$$W_{kk+1}(\underline{i} + \alpha_k) := \{(L_\bullet, L'_\bullet) : L'_k \xrightarrow{1} L_k \xrightarrow{i_{k+1}} L'_{k+1} \xrightarrow{1} L_{k+1}, \text{ and } L_j = L'_j \text{ for } j \neq k, k+1\}$$

inside $Y(\underline{i} + \alpha_k) \times Y(\underline{i} + 2\alpha_k + \alpha_{k+1})$. Finally, we need to precompose this with $\mathcal{E}_k \mathbf{1}_{\underline{i}}$. We find that the intersection $\pi_{12}^{-1}(W_k) \cap \pi_{23}^{-1}(W_{k+1})$ is of the expected dimension but has two smooth components A and B defined by

$$\begin{aligned} A &:= \{(L_\bullet, L'_\bullet, L''_\bullet) : L''_k \rightarrow L'_k \rightarrow L_k \rightarrow L''_{k+1} \rightarrow L_{k+1} = L'_{k+1}, \text{ and } L_j = L'_j = L''_j \text{ if } j \neq k, k+1\} \\ B &:= \{(L_\bullet, L'_\bullet, L''_\bullet) : L''_k \rightarrow L'_k \rightarrow L_k, L'_k \rightarrow L''_{k+1} \rightarrow L_{k+1} = L'_{k+1}, zL_{k+1} \subset L''_k, \\ &\quad \text{and } L_j = L'_j = L''_j \text{ if } j \neq k, k+1\}. \end{aligned}$$

Keeping track of line bundles we find that $(\mathcal{E}_k * \mathcal{E}_{k+1}) * \mathcal{E}_k \mathbf{1}_{\underline{i}} \cong \pi_{13*}(\mathcal{O}_{A \cup B} \otimes \mathcal{L})$ where

$$\mathcal{L} := \det(L_{k+2}/L_k)^{-1} \det(L'_k/L''_k) \det(L''_k/L_{k-1})^2 \{2i_k + i_{k+1} - 3\}.$$

Now $E := A \cap B$ is a divisor inside A and B . In fact, E is cut out of A by the map $z : \mathcal{L}_{k+1}/\mathcal{L}''_{k+1} \rightarrow \mathcal{L}'_k/\mathcal{L}''_k\{2\}$ and out of B by the map $\mathcal{L}_k/\mathcal{L}'_k \rightarrow \mathcal{L}_{k+1}/\mathcal{L}''_{k+1}$ induced by the inclusion $\mathcal{L}_k \hookrightarrow \mathcal{L}_{k+1}$. Using the standard exact triangle

$$\mathcal{O}_A(-E) \oplus \mathcal{O}_B(-E) \rightarrow \mathcal{O}_{A \cup B} \rightarrow \mathcal{O}_E$$

one obtains the exact triangle

$$(49) \quad \pi_{13*}(\mathcal{O}_A \otimes \mathcal{L}_A) \oplus \pi_{13*}(\mathcal{O}_B \otimes \mathcal{L}_B) \rightarrow \mathcal{E}_k * \mathcal{E}_{k+1} * \mathcal{E}_k \mathbf{1}_{\underline{i}} \rightarrow \pi_{13*}(\mathcal{O}_E \otimes \mathcal{L})$$

where

$$\begin{aligned} \mathcal{L}_A &:= \det(\mathcal{L}_{k+2}/\mathcal{L}_k)^{-1} \det(\mathcal{L}_{k+1}/\mathcal{L}''_{k+1}) \det(\mathcal{L}''_k/\mathcal{L}_{k-1})^2 \{2i_k + i_{k+1} - 5\}, \\ \mathcal{L}_B &:= \det(\mathcal{L}_k/\mathcal{L}''_k) \det(\mathcal{L}''_k/\mathcal{L}_{k-1})^2 \det(\mathcal{L}_{k+2}/\mathcal{L}_k)^{-1} \det(\mathcal{L}_{k+1}/\mathcal{L}''_{k+1})^{-1} \{2i_k + i_{k+1} - 3\}. \end{aligned}$$

Now $\pi_{13}(A) = W_{k(2)k+1}$, the map $\pi_{13}|_A : A \rightarrow W_{k(2)k+1}$ is generically one-to-one and \mathcal{L}_A is pulled back from $W_{k(2)k+1}$. Since A and $W_{k(2)k+1}$ are both smooth we have $\pi_{13*}(\mathcal{O}_A) \cong \mathcal{O}_{W_{k(2)k+1}}$ and hence

$$\begin{aligned} \pi_{13*}(\mathcal{O}_A \otimes \mathcal{L}_A) &\cong \mathcal{O}_{W_{k(2)k+1}} \otimes \det(\mathcal{L}_{k+2}/\mathcal{L}_{k+1})^{-1} \det(\mathcal{L}'_{k+1}/\mathcal{L}_k)^{-1} \det(\mathcal{L}'_k/\mathcal{L}_{k-1})^2 \{2i_k + i_{k+1} - 5\} \\ &\cong \mathcal{E}_k^{(2)} * \mathcal{E}_{k+1} \mathbf{1}_{\underline{i}}. \end{aligned}$$

A similar argument shows that $\pi_{13*}(\mathcal{O}_B \otimes \mathcal{L}_B) \cong \mathcal{E}_{k+1} * \mathcal{E}_k^{(2)} \mathbf{1}_\lambda$. Finally, $\pi_{13}|_E$ is a \mathbb{P}^1 bundle and \mathcal{L} restricts to $\mathcal{O}_{\mathbb{P}^1}(-1)$ on its fibres. Hence $\pi_{13*}(\mathcal{O}_E \otimes \mathcal{L}) = 0$. So (49) gives us the isomorphism in (48).

The second part of condition (viii) in [CaK3] is $\mathcal{E}_i * \mathcal{E}_j \cong \mathcal{E}_j * \mathcal{E}_i$ if $|i - j| > 1$. This follows from a direct computation of both sides (all intersections are of the expected dimension and push-forwards are one-to-one). Likewise for condition (ix) in [CaK3] which states that $\mathcal{E}_i * \mathcal{F}_j \cong \mathcal{F}_j * \mathcal{E}_i$ if $i \neq j$.

Condition (x) in [CaK3] states that \mathcal{E}_k and \mathcal{F}_k deform over $\tilde{Y}(\underline{i})|_B \times_B \tilde{Y}(\underline{i} + \alpha_k)|_B$ where $B = \alpha_k^\perp = \{\underline{x} : x_k = x_{k+1}\} \subset \mathbb{A}^{2N-1}$. This is clear since we can deform $W_k(\underline{i})$ to

$$\widetilde{W}_k(\underline{i}) := \{(L_\bullet, L'_\bullet, \underline{x}) : L_\bullet \subset L'_\bullet, \underline{x} \in \alpha_k^\perp\}$$

and the line bundles used to define \mathcal{E}_k and \mathcal{F}_k also deform. The condition $x_k = x_{k+1}$ is imposed on us because $(z - x_k)L'_k \subset L_{k-1}$ and $(z - x_{k+1})L_{k+1} \subset L_k$.

Condition (xi) in [CaK3] (the last condition) states that the cone \mathcal{E}_{kk+1} of the map $T_{ij} : \mathcal{E}_k * \mathcal{E}_{k+1}[-1]\{1\} \rightarrow \mathcal{E}_{k+1} * \mathcal{E}_k$ deforms over $\tilde{Y}(\underline{i})|_B \times_B \tilde{Y}(\underline{i} + \alpha_k + \alpha_{k+1})|_B$ where $B = (\alpha_k + \alpha_{k+1})^\perp = \{\underline{x} : x_k = x_{k+2}\} \subset \mathbb{A}^{2N-1}$. The key is to identify \mathcal{E}_{kk+1} explicitly. Just like in Lemma [CaK3, Lem. 3.10] one can show that

$$\mathcal{E}_{kk+1} \cong \mathcal{O}_{U_{kk+1}} \otimes \det(\mathcal{L}_{k+2}/\mathcal{L}'_k)^\vee \otimes \det(\mathcal{L}_{k+1}/\mathcal{L}_{k-1}) \{i_k + i_{k+1} - 1\}$$

where

$$U_{kk+1} := \{(L_\bullet, L'_\bullet) : L'_k \subset L_k, L'_{k+1} \subset L_{k+1}, L_j = L'_j \text{ for } j \neq k, k+1\} \subset Y(\underline{i}) \times Y(\underline{i} + \alpha_k + \alpha_{k+1}).$$

Then U_{kk+1} deforms to

$$\tilde{U}_{kk+1} := \{(L_\bullet, L'_\bullet, \underline{x}) : L'_k \subset L_k, L'_{k+1} \subset L_{k+1}, L_j = L'_j \text{ for } j \neq k, k+1, \underline{x} \in (\alpha_k + \alpha_{k+1})^\perp\}$$

and so do the line bundles used to define \mathcal{E}_{kk+1} . Again, the condition that $x_k = x_{k+2}$ is enforced by the following sequence of implications

$$(z - x_k)L_k \subset L_{k-1} \Rightarrow (z - x_k)L_k \subset L'_k \Rightarrow (z - x_k)L_{k+1} \subset L'_{k+1} \Rightarrow (z - x_k)L'_{k+2} \subset L'_{k+1}.$$

□

8.4. The categorical 2-representation $\mathcal{K}_{\text{Gr}, m}$. The 2-category $\mathcal{K}_{\text{Gr}, m}$ is defined as follows.

- The objects are $D(Y(\underline{0}, \underline{i}, \underline{m}))$ where $\underline{i} = (i_1, \dots, i_{2N})$ for some N and $\sum_k i_k$ is divisible by m .
- The 1-morphisms are all FM kernels and include $\mathcal{E}_k^{(r)} \mathbf{1}_{\underline{i}}$ and $\mathbf{1}_{\underline{i}} \mathcal{F}_k^{(r)}$ as defined in (46) and (47).
- The 2-morphisms are morphisms of FM kernels and include the maps

$$\theta_i : \mathcal{O}_{\Delta_* Y(\underline{0}, \underline{i}, \underline{m})} \rightarrow \mathcal{O}_{\Delta_* Y(\underline{0}, \underline{i}, \underline{m})}[2]\{-2\}$$

induced from the deformation $\tilde{Y}(\underline{i})$ of $Y(\underline{i})$ as outlined in section 8.2.

This data gives us a categorical 2-representation of \mathfrak{sl}_∞ . What remains is to identify the $U_q(\mathfrak{sl}_\infty)$ -module it categorifies.

The variety $Y(\underline{i})$ is an iterated Grassmannian bundle. Since the Grothendieck group of the Grassmannian $\mathbb{G}(i, m)$ has rank $\binom{m}{i}$ it follows that

$$\dim_{\mathbb{C}} K(Y(\underline{i})) = \prod_{k=1}^{2N} \binom{m}{i_k} \text{ where } \underline{i} = (i_1, \dots, i_{2N}).$$

Since a finite dimensional $U_q(\mathfrak{sl}_{2N})$ -module is uniquely determined by the dimensions of its weight spaces it follows that the data from Theorem 8.1 categorifies $\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$. Thus, letting $N \rightarrow \infty$ we end up with a categorification of the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$.

This completes the proof of Theorem 2.6

9. NAKAJIMA QUIVER VARIETIES AND THE 2-CATEGORY $\mathcal{K}_{Q,m}$

Since $\Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N})$ has commuting actions of $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_{2N})$ each \mathfrak{sl}_m weight space is preserved by the action of $U_q(\mathfrak{sl}_{2N})$. In particular, we can restrict the $U_q(\mathfrak{sl}_{2N})$ action to the zero weight space of \mathfrak{sl}_m , which is isomorphic to $(\Lambda_q^N(\mathbb{C}^{2N}))^{\otimes m}$.

Letting $N \rightarrow \infty$ we can likewise restrict the action of $U_q(\mathfrak{sl}_\infty)$ from $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ to $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$. In this section we explain how this smaller $U_q(\mathfrak{sl}_\infty)$ -module can be categorified using Nakajima quiver varieties. This leads to the same link invariants as the ones constructed above.

9.1. Categorification of $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$. In [CKL4] we defined geometric categorical \mathfrak{g} actions on Nakajima quiver varieties. We briefly recall this construction when $\mathfrak{g} = \mathfrak{sl}_{2N}$ following the notation in [CKL4].

Let $\Gamma = (I, E)$ be the finite Dynkin graph of \mathfrak{sl}_{2N} where I and E are the sets of vertices and edges. Given $\underline{v}, \underline{w} \in \mathbb{N}^I \cong \mathbb{N}^{2N-1}$ one can define the Nakajima quiver variety $\mathfrak{M}(\underline{v}, \underline{w})$ as a symplectic quotient (see section 3 of [CKL4]). Now, if we fix \underline{w} and allow \underline{v} to vary then $\mathfrak{M}(\underline{v}, \underline{w})$ corresponds to the weight space $\lambda := \sum_{i \in I} w_i \Lambda_i - \sum_{i \in I} v_i \alpha_i$. Certain correspondences between these varieties define kernels

$$\mathcal{E}_i^{(r)} \in D(\mathfrak{M}(\underline{v}, \underline{w})) \times \mathfrak{M}(\underline{v} - r\mathbf{e}_i, \underline{w}) \text{ and } \mathcal{F}_i^{(r)} \in D(\mathfrak{M}(\underline{v} - r\mathbf{e}_i, \underline{w}) \times \mathfrak{M}(\underline{v}, \underline{w}))$$

where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ has a 1 in position i . Moreover, varying the moment map condition also gives deformations $\widetilde{\mathfrak{M}}(\underline{v}, \underline{w})$.

Proposition 9.1 ([CKL4]). *The data above defines a geometric categorical \mathfrak{sl}_{2N} action on $\sqcup_{\underline{v}} \mathfrak{M}(\underline{v}, \underline{w})$ which categorifies the $U_q(\mathfrak{sl}_{2N})$ -module $\bigotimes_i V_{\Lambda_i}^{\otimes w_i} \cong \bigotimes_i \Lambda_q^i(\mathbb{C}^{2N})^{\otimes w_i}$. So, if we take*

$$\underline{w} = (0, \dots, 0, m, 0, \dots, 0) = (0^{N-1}, m, 0^{N-1}) \in \mathbb{N}^{2N-1}$$

then this categorifies the $U_q(\mathfrak{sl}_{2N})$ -module $\Lambda_q^N(\mathbb{C}^{2N})^{\otimes m}$.

Lemma 9.2. *The weight spaces of the $U_q(\mathfrak{sl}_{2N})$ -module $\Lambda_q^N(\mathbb{C}^{2N})^{\otimes m} \cong \bigoplus_{\underline{v}} K(\mathfrak{M}(\underline{v}, \underline{w}))$, where $\underline{w} = (0^{N-1}, m, 0^{N-1}) \in \mathbb{N}^{2N-1}$, are in bijection with $2N$ -tuples (i_1, \dots, i_{2N}) with $0 \leq i_1, \dots, i_{2N} \leq m$ and $\sum_{\ell} i_{\ell} = mN$. This bijection is given by*

$$\phi : \underline{v} \mapsto (0^N, m^N) + (v_1, -v_1 + v_2, -v_2 + v_3, \dots, -v_{2N-2} + v_{2N-1}, -v_{2N-1}).$$

Proof. Under the bijection in Lemma 7.1 the highest weight corresponds to $(0^N, m^N)$. In the case of Nakajima quiver varieties, the highest weight is given by $\underline{v} = \underline{0}$. The rest follows since $F_i : K(\mathfrak{M}(\underline{v}, \underline{w})) \rightarrow K(\mathfrak{M}(\underline{v} + \mathbf{e}_i, \underline{w}))$ while our other notation $F_i : V(\underline{i}) \rightarrow V(\underline{i} - \alpha_i)$ where $\alpha_i = (0, \dots, -1, 1, \dots, 0)$. \square

Now we take $N \rightarrow \infty$. More precisely, denote $\underline{v}' := (0, \underline{v}, 0) \in \mathbb{N}^{2N+1}$ and $\underline{w}' := (0, \underline{w}, 0) \in \mathbb{N}^{2N+1}$. By definition, $\mathfrak{M}(\underline{v}, \underline{w})$ and $\mathfrak{M}(\underline{v}', \underline{w}')$ are naturally isomorphic and likewise for the correspondences defining the geometric categorical $\mathfrak{sl}_{2(N+1)}$ action on $\sqcup_{\underline{v}'} \mathfrak{M}(\underline{v}', \underline{w}')$ and the \mathfrak{sl}_{2N} action on $\sqcup_{\underline{v}} \mathfrak{M}(\underline{v}, \underline{w})$. Hence we can take $N \rightarrow \infty$ to get a geometric categorical \mathfrak{sl}_∞ action on $\sqcup_{\underline{v}} \mathfrak{M}(\underline{v}, \underline{w})$ where $\underline{w} = (\dots, 0, m, 0, \dots)$. We denote the associated categorical 2-representation $\mathcal{K}_{Q,m}$. It categorifies the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$.

9.2. Subsequent link invariants. Consider a categorical 2-representation of \mathfrak{sl}_∞ which categorifies $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$. By Lemma 9.2 the nonzero weights $\mathcal{D}(\underline{i})$ are labeled by sequences $\underline{i} = (\dots, i_k, i_{k+1}, \dots)$ with $0 \leq i_k \leq m$, $i_k = 0$ for $k \ll 0$ and $i_k = m$ for $k \gg 0$ and where the sum of all $i_k \notin \{0, m\}$ is divisible by m . Then the same argument used to prove Theorem 2.4 implies the following.

Theorem 9.3. *Given a categorical 2-representation of \mathfrak{sl}_∞ which categorifies $\Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$ one can define functors for cups, caps and crossings as in section 2.3.3. When applied to a framed, oriented link K this gives an invariant in the form of a complex of functors*

$$\Psi_-(K) : \text{Kom}^-(\mathcal{D}(\underline{0}, \underline{m})) \rightarrow \text{Kom}^-(\mathcal{D}(\underline{0}, \underline{m}))$$

whose terms are direct sums of the identity functor (possibly with some grading shifts).

9.3. Relation to affine Grassmannians. The projection map

$$(50) \quad \Lambda_q^{mN}(\mathbb{C}^m \otimes \mathbb{C}^{2N}) \longrightarrow \Lambda_q^\infty(\mathbb{C}^{2\infty})^{\otimes m}$$

of $U_q(\mathfrak{sl}_\infty)$ -modules can be realized geometrically as follows. Recall that in section 8 we defined the twisted product varieties

$$Y(\underline{i}) = \{\mathbb{C}[z]^m = L_0 \subset L_1 \subset \dots \subset L_{2N} \subset z^{-2mN}\mathbb{C}[z]^m : zL_j \subset L_{j-1}, \dim(L_j/L_{j-1}) = i_j\}$$

where $\sum_k i_k = Nm$ (for notational convenience we have replaced $\mathbb{C}(z)^m$ with $z^{-2mN}\mathbb{C}[z]^m$ in the definition of $Y(\underline{i})$ above). This allows us to define the projection

$$Pr : z^{-2mN}\mathbb{C}[z]^m \cong z^{-2mN}\mathbb{C}[z] \otimes \mathbb{C}^m \longrightarrow z^{-N}\mathbb{C}[z] \otimes \mathbb{C}^m \cong z^{-N}\mathbb{C}[z]^m$$

as follows

$$Pr(z^k \otimes v) = \begin{cases} z^k \otimes v & \text{if } k \geq -N \\ 0 & \text{if } k < -N. \end{cases}$$

where $v \in \mathbb{C}^m$. Then we can define $U(\underline{i}) \subset Y(\underline{i})$ as the locus of points in $L_\bullet \in Y(\underline{i})$ such that $Pr(L_{2N}) = z^{-N}\mathbb{C}[z]^m$ or, equivalently, the locus of points where $\dim(Pr(L_{2N})/\mathbb{C}[z]^m) = Nm$. Since $\dim(L_{2N}/\mathbb{C}[z]^m) = \sum_k i_k = Nm$ it is not hard to see that $U(\underline{i}) \subset Y(\underline{i})$ is an open subscheme.

Using [MV] one can show that $U(\underline{i})$ is actually isomorphic to $\mathfrak{M}(\underline{v}, \underline{w})$ where $\underline{w} = (0^{N-1}, m, 0^{N-1})$ and $\phi(\underline{v}) = \underline{i}$ via the bijection ϕ from Lemma 9.2. Moreover, if we denote by u the natural inclusion of $\mathfrak{M}(\underline{v}, \underline{w})$ into $Y(\underline{i})$ then the restriction map $u^* : D(Y(\underline{i})) \rightarrow D(\mathfrak{M}(\underline{v}, \underline{w}))$ intertwines the geometric categorical \mathfrak{sl}_∞ actions on $\sqcup_{\underline{i}} Y(\underline{i})$ and $\sqcup_{\underline{v}} \mathfrak{M}(\underline{v}, \underline{w})$. In other words,

$$u^* \circ E_i^{(r)} \cong E_i^{(r)} \circ u^* \text{ and } u^* \circ F_i^{(r)} \cong F_i^{(r)} \circ u^*$$

so that u^* categorifies the projection map in (50). Another way of saying this is that u^* induces a 2-functor $\mathcal{K}_{\text{Gr}, m} \rightarrow \mathcal{K}_{\text{Q}, m}$ which categorifies (50).

Subsequently, the homological \mathfrak{sl}_m link invariants constructed using Theorem 2.4 and the affine Grassmannian are the same as those constructed (in the same way) using Theorem 9.3 and Nakajima quiver varieties.

10. SOME EXAMPLE COMPUTATIONS

To illustrate the constructions of clasps in this paper we will compute the cohomology of the unknot labeled by the \mathfrak{sl}_m representation $V_{2\Lambda_1} = \text{Sym}^2 V$ (where V is the standard m -dimensional representation of \mathfrak{sl}_m). This knot is illustrated in figure 5 where the P denotes the clasp P^- categorifying the composition $V \otimes V \xrightarrow{\pi} V_{2\Lambda_1} \xrightarrow{\iota} V \otimes V$.

In what follows we will assume we have a 2-representation of \mathfrak{sl}_∞ in the sense of [KL3, Rou1]. Perhaps the main motivation for performing these calculations is to illustrate how the computation of our link homologies can be performed entirely within the realm of this higher representation theory.

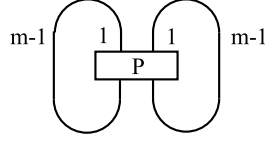


FIGURE 5.

The reader will note that our definition of a categorical 2-representation (from section 2.2) is only a simplified version of the definition in [KL3, Rou1]. For instance, it does not mention anything about KLR (a.k.a. quiver Hecke algebras) algebras. Our definition has the advantage of being easier to check in practice. For example, we do not know a direct way to check that the action from section 8 lifts to a 2-representation in the sense of [KL3, Rou1].

One way to deal with this issue is as follows. In [CKL2] we show that given a categorical 2-representation of \mathfrak{sl}_2 one can extend it so that the nilHecke algebra (which is the KLR algebra of \mathfrak{sl}_2) acts. In [Ca2] we will generalize this to any simply laced Kac-Moody algebra, thus showing that the KLR algebras also act in this case. Combining this result with [CLa] implies that a categorical 2-representation as defined in this paper lifts to a 2-representation in the sense of [KL3, Rou1].

Note however, that without the extra structure from [KL3] we still have a link invariant. It is just more difficult to compute.

10.1. Some notation. Very briefly, recall the diagrammatic notation from [KL3]. We will only deal with the \mathfrak{sl}_2 case where $E = E_1$ and $F = F_1$. An upward (resp. downward) pointing strand \uparrow (resp. \downarrow) denotes E (resp. F). Caps and cups denote adjunctions. A crossing denotes the nilHecke 2-morphism $T : EE \rightarrow EE\langle -2 \rangle$ while a solid dot \uparrow indicates the 2-morphism $X : E \rightarrow E\langle 2 \rangle$. By convention, 1-morphisms are composed horizontally going to the left while 2-morphisms are composed vertically going upwards.

We refer the reader to [KL3] for complete list of relations satisfied by such diagrams. One should not confuse such diagrams with link diagrams like the one in figure 5.

10.2. The clasp. First we need an explicit description of the clasp P^- . In this case it involves only two strands, so $\mathfrak{g} = \mathfrak{sl}_2$ and we have

$$\mathcal{D}(-2) \begin{matrix} \xrightarrow{E} \\ \xleftarrow{F} \end{matrix} \mathcal{D}(0) \begin{matrix} \xrightarrow{E} \\ \xleftarrow{F} \end{matrix} \mathcal{D}(2)$$

where $\mathcal{D}(0)$ categorifies $V \otimes V$ while $\mathcal{D}(-2)$ and $\mathcal{D}(2)$ categorify $\Lambda_q^2(V)$. Here we abbreviated E_1 and F_1 as E and F . The projector $P^- \mathbf{1}_0$ is given by $T_\omega^\infty \mathbf{1}_0 = \lim_{\ell \rightarrow \infty} T^{2\ell} \mathbf{1}_0$ which we now describe explicitly.

First, we have

$$(51) \quad T \mathbf{1}_0 \cong [EF \mathbf{1}_0 \langle -1 \rangle \xrightarrow{\quad \curvearrowright \quad} \mathbf{1}_0]$$

and squaring gives

$$T^2 \mathbf{1}_0 \cong \left[EFEF \mathbf{1}_0 \langle -2 \rangle \xrightarrow{(\uparrow \downarrow \curvearrowright, \curvearrowright \uparrow \downarrow)} EF \mathbf{1}_0 \langle -1 \rangle \oplus EF \mathbf{1}_0 \langle -1 \rangle \xrightarrow{(\curvearrowright, - \curvearrowright)} \mathbf{1}_0 \right]$$

Now $FE \mathbf{1}_{-2} \cong \mathbf{1}_{-2} \langle -1 \rangle \oplus \mathbf{1}_{-2} \langle 1 \rangle$ where the isomorphisms are given by

$$(52) \quad \curvearrowright \text{ and } \curvearrowleft \text{ with inverses } \curvearrowright - \curvearrowleft \text{ and } \curvearrowleft \text{ with inverses } \curvearrowright$$

Using these we find that $T^2 \mathbf{1}_0$ is isomorphic to the complex

$$\begin{array}{ccccc}
 \text{EF}\mathbf{1}_0\langle -3 \rangle & \xrightarrow{\uparrow\downarrow} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{\quad} & \mathbf{1}_0 \\
 & \searrow \text{id} & \nearrow & & \\
 \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{\uparrow\downarrow} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{-} & \mathbf{1}_0
 \end{array}$$

Using the cancellation on the two terms in the bottom row we end up with

$$(53) \quad T^2 \mathbf{1}_0 \cong [\text{EF}\mathbf{1}_0\langle -3 \rangle \xrightarrow{\uparrow\downarrow - \uparrow\downarrow} \text{EF}\mathbf{1}_0\langle -1 \rangle \xrightarrow{\quad} \mathbf{1}_0].$$

Now composing (51) with (53) gives us that $T^3 \mathbf{1}_0$ is isomorphic to

$$\begin{array}{ccccc}
 & & \text{EF}\mathbf{1}_0\langle -3 \rangle & \xrightarrow{\uparrow\downarrow - \uparrow\downarrow} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{\quad} & \mathbf{1}_0 \\
 & \nearrow & & \nearrow & & & \\
 \text{EFEF}\mathbf{1}_0\langle -4 \rangle & \xrightarrow{\uparrow\downarrow\uparrow\downarrow - \uparrow\downarrow\uparrow\downarrow} & \text{EFEF}\mathbf{1}_0\langle -2 \rangle & \xrightarrow{\uparrow\downarrow} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{-} & \mathbf{1}_0
 \end{array}$$

Using (52) again we can expand to obtain

$$\begin{array}{ccccc}
 & \uparrow\downarrow & \text{EF}\mathbf{1}_0\langle -3 \rangle & \xrightarrow{-\uparrow\downarrow + \uparrow\downarrow} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{\quad} & \mathbf{1}_0 \\
 & \nearrow \text{id} & & \nearrow \text{id} & & & \\
 \text{EF}\mathbf{1}_0\langle -5 \rangle & \xrightarrow{\alpha} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{-\text{id}} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{-} & \mathbf{1}_0 \\
 & \searrow \beta & \nearrow & \nearrow & & & \\
 \text{EF}\mathbf{1}_0\langle -3 \rangle & \xrightarrow{\text{id}} & \text{EF}\mathbf{1}_0\langle -3 \rangle & \xrightarrow{\uparrow\downarrow} & \text{EF}\mathbf{1}_0\langle -1 \rangle & \xrightarrow{\quad} & \mathbf{1}_0
 \end{array}$$

where $\alpha = -\uparrow \circlearrowleft^{-1} \downarrow$ and $\beta = -\uparrow\downarrow + \uparrow \circlearrowleft^2 \downarrow$. Note that to end up with this complex we used several relations such as the one relating clockwise and counterclockwise bubbles [KL3, Eq. 3.7] and the bubble slide relations [KL3, Prop. 3.3]. In the end, using the cancellation lemma we end up with

$$(54) \quad T^3 \mathbf{1}_0 \cong [\text{EF}\mathbf{1}_0\langle -5 \rangle \xrightarrow{\uparrow\downarrow + \uparrow\downarrow - \uparrow \circlearrowleft^2 \downarrow} \text{EF}\mathbf{1}_0\langle -3 \rangle \xrightarrow{\uparrow\downarrow - \uparrow\downarrow} \text{EF}\mathbf{1}_0\langle -1 \rangle \xrightarrow{\quad} \mathbf{1}_0].$$

The fact that the composition of the first two maps above is zero is a consequence of

$$\uparrow\downarrow^2 - \uparrow \circlearrowleft^2 \downarrow + \uparrow \circlearrowleft^{-1} \downarrow = 0 = \uparrow\downarrow^2 - \uparrow \circlearrowleft^2 \downarrow + \uparrow \circlearrowleft^{-1} \downarrow$$

which follows from relation [KL3, Eq. 3.5].

Now, to simplify our computation, we will hereon assume that any positive degree endomorphism of $\mathbf{1}_2$ is zero. This is the case, for instance, in our categorical 2-representations $\mathcal{K}_{\text{Gr},m}$ and $\mathcal{K}_{\text{Q},m}$. Then, iterating the argument above we find that

$$(55) \quad T^{2n} \mathbf{1}_0 \cong [\text{EF}\mathbf{1}_0\langle -2n-1 \rangle \rightarrow \text{EF}\mathbf{1}_0\langle -2n+1 \rangle \rightarrow \cdots \rightarrow \text{EF}\mathbf{1}_0\langle -1 \rangle \rightarrow \mathbf{1}_0]$$

where the maps alternate between $\uparrow\downarrow - \downarrow\uparrow$ and $\uparrow\downarrow + \downarrow\uparrow$. Thus $P^- \mathbf{1}_0 = \lim_{\ell \rightarrow \infty} T^{2\ell} \mathbf{1}_0$ is the obvious limit of this complex.

10.3. Computation of the cohomology. In terms of our categorical 2-representation, the homology of the knot in figure 5 is given by the composition

$$(56) \quad E_3 F_1 P_2^- E_1 F_3 : \text{Kom}^-(\mathcal{D}(\underline{0}, m, 0, 0, m, \underline{m})) \rightarrow \text{Kom}^-(\mathcal{D}(\underline{0}, m, 0, 0, m, \underline{m}))$$

where $P_2^- = T_2^\infty$ is the infinite complex

$$(57) \quad [\cdots \rightarrow E_2 F_2 \mathbf{1}_{\underline{i}} \langle -2n-1 \rangle \rightarrow E_2 F_2 \mathbf{1}_{\underline{i}} \langle -2n+1 \rangle \rightarrow \cdots \rightarrow E_2 F_2 \mathbf{1}_{\underline{i}} \langle -3 \rangle \rightarrow E_2 F_2 \mathbf{1}_{\underline{i}} \langle -1 \rangle \rightarrow \mathbf{1}_{\underline{i}}]$$

where $\underline{i} = (\underline{0}, m-1, 1, 1, m-1, \underline{m})$. Now,

$$\begin{aligned} E_3 F_1 (E_2 F_2) E_1 F_3 \mathbf{1}_{\underline{k}} &\cong E_3 E_2 F_1 E_1 \mathbf{1}_{(\underline{0}, m, 1, 0, m-1, \underline{m})} F_2 F_3 \mathbf{1}_{\underline{k}} \\ &\cong \oplus_{[m-1]} E_3 E_2 F_2 \mathbf{1}_{(\underline{0}, m, 0, 1, m-1, \underline{m})} F_3 \mathbf{1}_{\underline{k}} \\ &\cong \oplus_{[m-1]} E_3 F_3 \mathbf{1}_{\underline{k}} \\ &\cong \oplus_{[m]} \oplus_{[m-1]} \mathbf{1}_{\underline{k}} \end{aligned}$$

where $\underline{k} = (\underline{0}, m, 0, 0, m, \underline{m})$. Subsequently, the composition in (56) simplifies to give a complex

$$(58) \quad \left[\cdots \rightarrow \bigoplus_{[m][m-1]} \mathbf{1}_{\underline{k}} \langle -2n-1 \rangle \rightarrow \bigoplus_{[m][m-1]} \mathbf{1}_{\underline{k}} \langle -2n+1 \rangle \rightarrow \cdots \rightarrow \bigoplus_{[m][m-1]} \mathbf{1}_{\underline{k}} \langle -1 \rangle \rightarrow \bigoplus_{[m][m]} \mathbf{1}_{\underline{k}} \right]$$

where the right most term is a consequence of the fact that

$$E_3 F_1 E_1 F_3 \mathbf{1}_{\underline{k}} \cong E_3 F_3 E_1 F_1 \mathbf{1}_{\underline{k}} \cong \bigoplus_{[m][m]} \mathbf{1}_{\underline{k}}.$$

It remains to identify the differentials in (58).

Lemma 10.1. *Let us denote $A := \bigoplus_{[m-1]} \mathbf{1}_{\underline{k}}$. The right most map in (58) is injective. After that, going to our left, the maps alternate between the zero map and some $\gamma : \bigoplus_{[m]} A \rightarrow \bigoplus_{[m]} A \langle 2 \rangle$ which is an isomorphism between the $m-1$ summands $A \langle -m+3 \rangle, \dots, A \langle m-1 \rangle$ on either side.*

Proof. First we show that the right most map is injective. For any $a, b \in \mathbb{N}$, we can include $\mathbf{1}_{\underline{k}}$ into $E_3 F_1 E_2 F_2 E_1 F_3 \mathbf{1}_{\underline{k}}$ as the composition depicted in the lower half (the part below the dashed line) of the diagram on the left in figure 6. As we vary $0 \leq a \leq m-1$ and $0 \leq b \leq m-2$ we pick out every summand $\mathbf{1}_{\underline{k}}$ inside $E_3 F_1 E_2 F_2 E_1 F_3 \mathbf{1}_{\underline{k}}$.

Now, the rightmost differential in (58) is given by the adjunction map. To show that it induces an injection it suffices to show that the composition on the left in figure 6 is some nonzero multiple of the identity, for any $0 \leq a \leq m-1$ and $0 \leq b \leq m-2$. To see this, one can move the circle labeled 1 inside the circle labeled 2 and, using the fact that the dots move through up-down crossings, obtain the composition encoded by the right hand diagram of figure 6.

The circle in the middle is now equal to the identity and can be removed. Note that the number of dots one needs on a circle for it to equal the identity depends on the weight space and is given by relation (iii) on page 22 of [KL3]. For the same reason, the circle labeled 2 is also equal to the identity and likewise for the final remaining circle labeled by 3. This proves that the composition is equal to the identity.

Now we need to understand the remaining maps in the complex (58). First notice that, for degree reasons, any map $\bigoplus_{[m][m-1]} \mathbf{1}_{\underline{k}} \langle -2n-1 \rangle \rightarrow \bigoplus_{[m][m-1]} \mathbf{1}_{\underline{k}} \langle -2n+1 \rangle$ induces an isomorphism between at most $(m-1)^2$ summands of the form $\mathbf{1}_{\underline{k}}$. Thus, it suffices to show that γ induces an isomorphism on $(m-1)^2$ such summands.

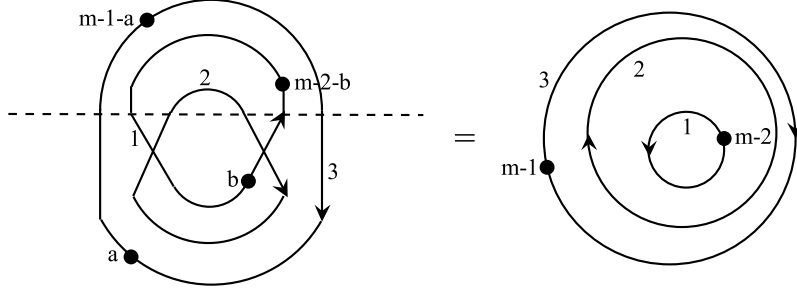


FIGURE 6.

Recall that the differentials in (57) alternate between $\uparrow\downarrow - \downarrow\uparrow$ and $\uparrow\downarrow + \downarrow\uparrow$. Now $\uparrow\downarrow$ and $\downarrow\uparrow$ induce the two compositions on the left and middle diagrams in figure 7 (they occur in between the two dashed lines). These two compositions are clearly the same because we can move the dot around to the other side. So $\uparrow\downarrow - \downarrow\uparrow$ induces zero while $\uparrow\downarrow + \downarrow\uparrow$ induces (two times) the left hand composition in figure 7.

Now, as we did above, we can move the circle labeled 1 into the middle and then remove it since it is equal to the identity. Thus we are left with the composition on the right side in figure 7. The middle circle labeled 2 is no longer equal to the identity but we can try to move it outside of the larger circle. In so doing we end up with two diagrams (this uses relation (3.14) from [KL3]). In the first, the circle is outside and becomes zero. In the other the circle disappears but the outer circle (labeled 3) picks up another dot (for a total of $m - 1$ dots) and is thus equal to the identity.

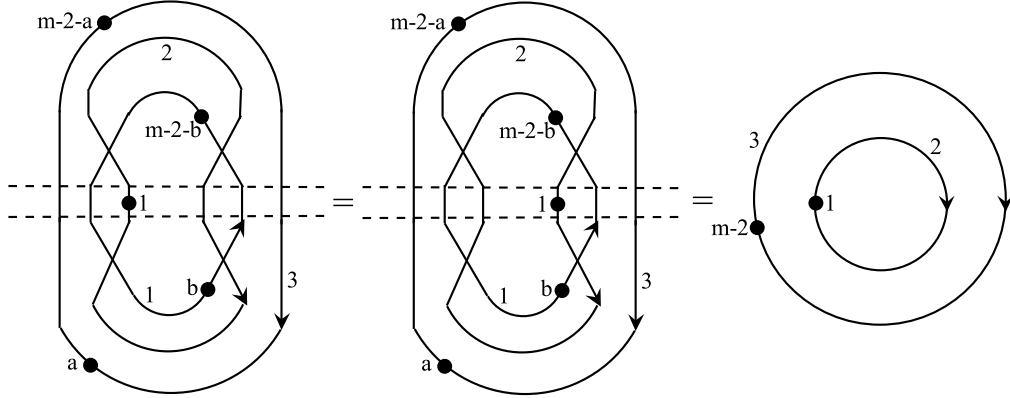


FIGURE 7.

To summarize, the composition in (the left or middle diagrams of) figure 7 is equal to the identity whenever $0 \leq a, b \leq m - 2$. In other words γ induces an isomorphism between $(m - 1)^2$ summands of the form $\mathbf{1}_{\underline{k}}$, which is what we needed to show. \square

Since, for \underline{k} as above, we assumed $\text{Hom}(\mathbf{1}_{\underline{k}}, \mathbf{1}_{\underline{k}}\langle \ell \rangle) = 0$ if $\ell \neq 0$ we can replace $\mathbf{1}_{\underline{k}}$ by \mathbb{C} in (58) to obtain a complex of graded vector spaces.

Corollary 10.2. *Denote by $\chi_{q,t}(V_{2\Lambda_1})$ the graded Poincaré polynomial of the \mathfrak{sl}_m homology of the unknot labeled by $V_{2\Lambda_1}$. Then*

$$(59) \quad \chi_{q,t}(V_{2\Lambda_1}) = 1 + [m-1] \frac{q^{-m} + t^3 q^{m+4}}{1 - t^2 q^4}.$$

Remark 10.3. Recall that, by convention, q corresponds to the grading shift $\langle -1 \rangle$ while t corresponds to $[1]$ (a downward shift by one in cohomology).

Proof. We still use the notation $A = \bigoplus_{[m-1]} \mathbb{C}$. Notice that the kernel and cokernel of γ are $A\langle -m+1 \rangle$ and $A\langle m+1 \rangle$ respectively. So, after using γ to cancel out as many of the summands A as possible, we are left with the following complex

$$(60) \quad \dots \xrightarrow{0} A\langle -m-8 \rangle \rightarrow A\langle m-8 \rangle \xrightarrow{0} A\langle -m-4 \rangle \rightarrow A\langle m-4 \rangle \xrightarrow{0} 0 \rightarrow \bigoplus_{[m]} \mathbb{C}\langle m-1 \rangle.$$

The maps indicated above are zero because, by Lemma 10.1, they were already zero in (58). The remaining maps must also be zero for degree reasons. Thus the graded Poincaré polynomial is

$$q^{-m+1}[m] + [m-1] (q^{-m+4}t^2 + q^{m+4}t^3 + q^{-m+8}t^4 + q^{m+8}t^5 + \dots)$$

which simplifies to give (59). \square

It is easy to check that $\chi_{q,-1}(V_{2\Lambda_1}) = \frac{[m][m+1]}{[2]}$ which, as expected, is the quantum dimension of $V_{2\Lambda_1} = \text{Sym}^2 V$.

10.3.1. *The cases $m = 2, 3$.* When $m = 2$, Corollary 10.2 gives $\chi_{q,t}(V_{2\Lambda_1}) = 1 + \frac{q^{-2} + t^3 q^6}{1 - t^2 q^4}$. This invariant was also computed in [CoK] (see section 4.3.1). Their homology is supported in positive degree so their Poincaré polynomial is actually $1 + \frac{q^{-2} + t^{-3} q^6}{1 - t^{-2} q^4}$, obtained by replacing t with t^{-1} (one should also take $q \mapsto q^{-1}$ but this is not necessary since they use the convention that $\langle 1 \rangle \leftrightarrow q$ which is opposite from ours).

When $m = 3$, Corollary 10.2 gives $\chi_{q,t}(V_{2\Lambda_1}) = 1 + [2] \frac{q^{-3} + t^3 q^7}{1 - t^2 q^4}$. This was also computed in [Ros, Example 5.4] (the calculation in the first arXiv version of that paper contained a small typo).

10.3.2. *Working over \mathbb{Z} .* If one is more careful, the calculation above can be done over \mathbb{Z} rather than \mathbb{C} . In this case $A = \bigoplus_{[m-1]} \mathbb{Z}$ and the map γ in Lemma 10.1 is multiplication by 2 between the $m-1$ summands $A\langle -m+3 \rangle, \dots, A\langle m-1 \rangle$. Subsequently, the \mathbb{Z} -rank of the homology is still given by (59) but there is also $\mathbb{Z}/2\mathbb{Z}$ torsion. This torsion is isomorphic to

$$\bigoplus_{d \geq 1} \bigoplus_{[m-1][m-1]} \mathbb{Z}/2\mathbb{Z}\langle -4d \rangle [2d]$$

(i.e. it occurs in cohomological degrees $-2, -4, -6, \dots$). When $m = 2$ this torsion was also computed in [CoK] at the end of section 4.3.1 (in their notation, it corresponds to using the parameter $\alpha = 0$).

10.3.3. *Torus links.* Cutting off the calculations above also computes the invariants of torus links. For example, consider the link in figure 5 where P is replaced by n crossings (i.e. T^n). This gives the $(2, n)$ torus link. Then the associated invariant is calculated by the complex in (60) where we chop off everything after the $(n-1)$ th term $A\langle \cdot \rangle$. So, for example, if n is odd then the Poincaré polynomial of the $(2, n)$ torus link becomes

$$1 + [m-1] \frac{q^{-m} + t^3 q^{m+4}}{1 - t^2 q^4} - [m-1] \frac{(tq^2)^{n+1} (q^m t + q^{-m})}{1 - t^2 q^4}.$$

When $m = 2$ this homology was originally computed by Khovanov in [Kh1, Prop. 26].

10.4. Another computation: the homology of the unknot labeled by $\Lambda_1 + \Lambda_{m-1}$. One can similarly try to compute the homology of the unknot labeled by the adjoint representation of \mathfrak{sl}_m (namely $V_{\Lambda_1 + \Lambda_{m-1}}$). In this case we use the same knot as in figure 5 but relabel the middle two strands $1, m-1$ instead of $1, 1$. Then the composition we need to compute is

$$F_3 F_1 P_2^- E_1 E_3 : \text{Kom}^-(\mathcal{D}(\underline{0}, m, 0, m, 0, \underline{m})) \rightarrow \text{Kom}^-(\mathcal{D}(\underline{0}, m, 0, m, 0, \underline{m}))$$

where $P_2^- = T_2^\infty$ is now the infinite complex

$$[\cdots \rightarrow F_2 E_2 \mathbf{1}_{\underline{i}} \langle -m(2n-1) - 1 \rangle \rightarrow F_2 E_2 \mathbf{1}_{\underline{i}} \langle -m(2n-1) + 1 \rangle \rightarrow \cdots \rightarrow F_2 E_2 \mathbf{1}_{\underline{i}} \langle -m + 1 \rangle \rightarrow \mathbf{1}_{\underline{i}}].$$

Here $\underline{i} = (\underline{0}, m-1, 1, m-1, 1, \underline{m})$ and, based on some calculations similar to the ones in the last section, it seems the maps alternate between

$$(61) \quad \downarrow \uparrow - \downarrow \uparrow \quad \text{and} \quad \sum_{0 \leq j \leq m-1} \downarrow_j \uparrow_{m-1-j}$$

although we have not checked this completely.

Now, one can show that $F_3 F_1 F_2 E_2 E_1 E_3 \mathbf{1}_{\underline{k}} \cong \oplus_{[m]} \mathbf{1}_{\underline{k}}$ where $\underline{k} = (\underline{0}, m, 0, m, 0, \underline{m})$. Thus we end up with a complex

$$\left[\cdots \rightarrow \bigoplus_{[m]} \mathbf{1}_{\underline{k}} \langle -m(2n-1) - 1 \rangle \rightarrow \bigoplus_{[m]} \mathbf{1}_{\underline{k}} \langle -m(2n-1) + 1 \rangle \rightarrow \cdots \rightarrow \bigoplus_{[m]} \mathbf{1}_{\underline{k}} \langle -m + 1 \rangle \rightarrow \bigoplus_{[m][m]} \mathbf{1}_{\underline{k}} \right].$$

The right hand map is again injective. The rest of the maps alternate between being zero and being an isomorphism on exactly one summand of the form $\mathbf{1}_{\underline{k}}$. After cancelling out, this gives a complex of the form

$$\left[\cdots \xrightarrow{0} \bigoplus_{[m-1]} \mathbb{C} \langle -5m \rangle \rightarrow \bigoplus_{[m-1]} \mathbb{C} \langle -3m \rangle \xrightarrow{0} \bigoplus_{[m-1]} \mathbb{C} \langle -3m \rangle \rightarrow \bigoplus_{[m-1]} \mathbb{C} \langle -m \rangle \xrightarrow{0} 0 \rightarrow \bigoplus_{[m][m-1]} \mathbb{C} \langle 1 \rangle \right].$$

Thus, we arrive at the Poincaré polynomial

$$\chi_{q,t}(V_{\Lambda_1 + \Lambda_{m-1}}) = [m-1] \left([m-1] + \frac{q^{-m} + t^3 q^{3m}}{1 - t^2 q^{2m}} \right).$$

A little calculation show that $\chi_{q,-1}(V_{\Lambda_1 + \Lambda_{m-1}}) = [m]^2 - 1$ (when $m = 3$ this was also computed in [Ros, Example 5.5]). Moreover, if we carefully redo the computation above over \mathbb{Z} then there will also be $\mathbb{Z}/m\mathbb{Z}$ -torsion. This torsion is isomorphic to $\bigoplus_{d \geq 1} \mathbb{Z}/m\mathbb{Z} \langle -2md \rangle [2d]$.

11. FURTHER REMARKS

11.1. The clasp P^+ and the associated link homology $\mathcal{H}_+^{i,j}(K)$. Recall that in section 7.5 we defined the complex $\bar{\Psi}_-(K)$ whose homology gives the link invariant $\mathcal{H}_-^{i,j}(K)$. This involved the projector $P^- \in \text{Kom}_*(\mathcal{K})$ defined as $\lim_{\ell \rightarrow \infty} T_\omega^{2\ell}$. One can equally well define the functor

$$P^+ := \lim_{\ell \rightarrow \infty} T_\omega^{-2\ell} \in \text{Kom}_*^+(\mathcal{K})$$

where Kom_*^+ is the analogue of Kom_*^- consisting of complexes that are bounded below (see section 3.3.3). The same argument gives a complex $\bar{\Psi}_+(K)$ whose homology $\mathcal{H}_+^{i,j}(K)$ is also a link invariant. In this section we explain in what sense $\mathcal{H}_-^{i,j}(K)$ and $\mathcal{H}_+^{i,j}(K)$ are dual to each other.

Remark 11.1. Note that P^- is defined as a direct limit of $T_\omega^{2\ell}$. In contrast, P^+ is defined as an *inverse* limit of $T_\omega^{-2\ell}$.

11.1.1. *Duality.* Consider a categorical 2-representation \mathcal{K} of \mathfrak{sl}_∞ and suppose each weight space $\mathcal{D}(\lambda)$ has a duality functor $\mathbb{D} : \mathcal{D}(\lambda) \rightarrow \mathcal{D}(\lambda)$ satisfying the following properties:

- (i) \mathbb{D} is contravariant and $\mathbb{D}(\mathcal{A}\langle 1 \rangle) = \mathbb{D}(\mathcal{A})\langle -1 \rangle$ for any object $\mathcal{A} \in \mathcal{D}(\lambda)$,
- (ii) $\mathbb{D} \circ \mathbf{E}_i \mathbf{1}_\lambda \cong \mathbf{F}_i \circ \mathbb{D} \langle \langle \lambda, \alpha_i \rangle + 1 \rangle$ and $\mathbb{D} \circ \mathbb{D} \cong \text{id}$.

Now, assuming all the 1-morphisms in \mathcal{K} admit adjoints we can extend this to a functor on the 2-representation $\oplus_\lambda \mathcal{D}(\lambda)$ as follows:

- $\mathbb{D}(\mathcal{D}(\lambda)) := \mathcal{D}(-\lambda)$ (i.e. $\mathcal{D}(\lambda)_{\text{new}} := \mathcal{D}(-\lambda)_{\text{old}}$),
- \mathbb{D} acts on 1-morphisms by taking their right adjoint. Thus it is contravariant with

$$\mathbb{D}(\mathbf{E}_i \mathbf{1}_\lambda) := (\mathbf{E}_i \mathbf{1}_\lambda)_R = \mathbf{F}_i \langle \langle \lambda, \alpha_i \rangle + 1 \rangle \text{ and } \mathbb{D}(\mathbf{1}_\lambda \mathbf{F}_i) := (\mathbf{1}_\lambda \mathbf{F}_i)_R = \mathbf{E}_i \langle -\langle \lambda, \alpha_i \rangle - 1 \rangle,$$
- \mathbb{D} is contravariant on 2-morphisms with $\mathbb{D}(\theta_i) := \theta_i$.

The fact that \mathbb{D} is contravariant means that it extends to $\text{Kom}_*^-(\mathcal{K})$ with $\mathbb{D}(\text{Kom}_*^-(\mathcal{K})) = \text{Kom}_*^+(\mathbb{D}(\mathcal{K}))$. Moreover, the representations categorified by \mathcal{K} and $\mathbb{D}(\mathcal{K})$ are dual (i.e. $K(\mathbb{D}(\mathcal{K})) \cong K(\mathcal{K})^\vee$).

Example. Consider the categorical 2-representation from section 8 where $\mathcal{D}(\lambda) = D(Y(\underline{i}))$ are used to categorify $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$. We take $\mathbb{D} : D(Y(\underline{i})) \rightarrow D(Y(\underline{i}))$ to be the Grothendieck-Verdier dualizing functor

$$\mathcal{A} \mapsto \mathcal{A}^\vee \otimes \omega_{Y(\underline{i})}[\dim Y(\underline{i})] \text{ for any } \mathcal{A} \in D(Y(\underline{i})).$$

Condition (ii) above holds because for any two kernels $\mathcal{P}_1, \mathcal{P}_2$ we have $(\mathcal{P}_2 * \mathcal{P}_1)_R \cong (\mathcal{P}_1)_R * (\mathcal{P}_2)_R$. Thus

$$\begin{aligned} \mathbb{D}(\mathbf{E}_i(\mathcal{A})) &= (\mathbf{E}_i(\mathcal{A}))^\vee \otimes \omega_{Y(\underline{i}')}[\dim Y(\underline{i}')] \\ &= (\mathcal{E}_i * \mathcal{A})_R \\ &\cong (\mathcal{A})_R * (\mathcal{E}_i \mathbf{1}_\lambda)_R \\ &= (\mathcal{A})_R * \mathbf{F}_i \langle \langle \lambda, \alpha_i \rangle + 1 \rangle \\ &= \mathbf{F}_i(\mathcal{A}^\vee \otimes \omega_{Y(\underline{i})}[\dim Y(\underline{i})]) \langle \langle \lambda, \alpha_i \rangle + 1 \rangle \\ &= \mathbf{F}_i(\mathbb{D}(\mathcal{A})) \langle \langle \lambda, \alpha_i \rangle + 1 \rangle \end{aligned}$$

where $\mathcal{E}_i \in D(Y(\underline{i}) \times Y(\underline{i}'))$ is the sheaf that induces \mathbf{E}_i and we think of \mathcal{A} as a kernel on $Y(\underline{i}) \times pt$. The map $\lambda \mapsto -\lambda$ corresponds to $\underline{i} \mapsto \underline{m} - \underline{i} := (\dots, m - i_{k-1}, m - i_k, m - i_{k+1}, \dots)$.

Remark 11.2. All along in this paper we have been thinking of the $U_q(\mathfrak{sl}_\infty)$ -module $\Lambda_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ as a highest weight representation where the highest weight corresponds to $(\dots, 0, 0, m, m, \dots)$. Its dual is naturally a lowest weight representation with lowest weight $(\dots, m, m, 0, 0, \dots)$. The duality \mathbb{D} interchanges the 2-categories of highest and lowest weight representations. If one ignores the extra categorical structure which $\mathcal{D}(\lambda)$ carries then such a duality has been considered before (see, for example, [Rou2, Remark 4.9]).

Let us see how \mathbb{D} acts on the associated tangle invariants. First, up to some shift, it flips cups and caps. To see this, note that a cup and a cap are defined by maps

$$\mathbf{F}_i^{(k)} : \mathcal{D}(\dots, 0, m, \dots) \rightarrow \mathcal{D}(\dots, k, m - k, \dots) \text{ and } \mathbf{E}_i^{(k)} : \mathcal{D}(\dots, k, m - k, \dots) \rightarrow \mathcal{D}(\dots, 0, m, \dots).$$

Then

$$(62) \quad \mathbb{D} \circ \mathbf{E}_i^{(k)} \cong (\mathbf{E}_i^{(k)})_R \circ \mathbb{D} \cong \mathbf{F}_i^{(k)} \circ \mathbb{D} \langle m - k \rangle \text{ and } \mathbb{D} \circ \mathbf{F}_i^{(k)} \cong (\mathbf{F}_i^{(k)})_R \circ \mathbb{D} \cong \mathbf{E}_i^{(k)} \circ \mathbb{D} \langle -m + k \rangle.$$

Second, it exchanges positive and negative crossings because $(\mathbf{T}_i)_R \cong \mathbf{T}_i^{-1}$. Finally, since $\mathbf{P}^- = \lim_{\ell \rightarrow \infty} \mathbf{T}_\omega^{2\ell}$ it also interchanges \mathbf{P}^- and \mathbf{P}^+ .

Proposition 11.3. *Let K be a framed, oriented link and denote by $K^!$ its mirror. Then $\overline{\Psi}_+(K) \cong \overline{\Psi}_-(K^!)^\vee$ and subsequently $\mathcal{H}_+^{i,j}(K) \cong \mathcal{H}_-^{-i,-j}(K^!)$.*

Proof. The homology $\mathcal{H}^{i,j}_-(K)$ is computed from $\Psi_-(K)$. Now consider $\mathbb{D} \circ \Psi_-(K)$. Moving \mathbb{D} to the right has the effect of flipping K about a horizontal line, interchanging positive and negative crossings and exchanging all projectors P^- with P^+ (at least up to some overall grading shift).

Now, flipping K gives an equivalent link while exchanging over and under crossings amounts to replacing K by its mirror $K^!$. Moreover, in the end there is no overall grading shift since, in a link, caps and cups always come in pairs and so the shifts in (62) actually cancel out.

Thus we end up with $\mathbb{D} \circ \Psi_-(K) \cong \Psi_+(K^!) \circ \mathbb{D}$ and the result follows. \square

Remark 11.4. If K does not contain any clasps (*i.e.* its strands are labeled only by fundamental representations) then $\mathcal{H}^{i,j}_+(K) \cong \mathcal{H}^{i,j}_-(K)$ and there is only one homology, which we denote simply $\mathcal{H}^{i,j}(K)$. Then Proposition 11.3 implies that $\mathcal{H}^{i,j}(K) \cong \mathcal{H}^{-i,-j}(K^!)$. When $m = 2$ (*i.e.* in the case of Khovanov homology) this fact was originally observed in [Kh1, Cor. 11].

11.2. Triangulated 2-representations. Let us consider the categorical 2-representation $\mathcal{K}_{\text{Gr},m}$ from section 8. This 2-category is actually triangulated (in the sense of section 3.1.3) although in this paper we ignored this extra structure. The grading $\langle 1 \rangle$ is equal to $[1]\{-1\}$ where $[\cdot]$ is the cohomological grading and $\{\cdot\}$ is the grading corresponding to the \mathbb{C}^\times action on the varieties.

Now consider larger the 2-category $\mathcal{K}_{\text{Gr},m}^+$ where we allow complexes of sheaves and FM kernels which are unbounded above (one can also allow unbounded below complexes which gives $\mathcal{K}_{\text{Gr},m}^-$). The 2-category $\mathcal{K}_{\text{Gr},m}^+$ should *not* be confused with $\text{Kom}^+(\mathcal{K}_{\text{Gr},m})$.

Now, consider a complex $(A_\bullet, f_\bullet) = A_n \xrightarrow{f_n} A_{n-1} \rightarrow \cdots \xrightarrow{f_1} A_0$ of 1-morphisms. This is naturally an object in $\text{Kom}(\mathcal{K}_{\text{Gr},m})$, but since $\mathcal{K}_{\text{Gr},m}$ is triangulated, we can try to take a convolution $C(A_\bullet)$ of A_\bullet . Let us briefly recall this concept (see also [GM] section IV, exercise 1).

If $n = 1$ then $C(A_\bullet)$ is just the cone of $f_1 : A_1 \rightarrow A_0$. If $n > 1$ it is an iterated cone. More precisely, a (right) *convolution* of (A_\bullet, f_\bullet) is a 1-morphism B such that there exist

- (i) 1-morphisms $A_0 = B_0, B_1, \dots, B_{n-1}, B_n = B$ and
- (ii) 2-morphisms $g_i : B_i[-i] \rightarrow A_i, h_i : A_i \rightarrow B_{i-1}[-(i-1)]$ (with $h_0 = id$)

such that for each i

$$(63) \quad B_i[-i] \xrightarrow{g_i} A_i \xrightarrow{h_i} B_{i-1}[-(i-1)]$$

is a distinguished triangle and $g_{i-1} \circ h_i = f_i$. Such a collection of data is called a *Postnikov system*.

In general a complex may not have a convolution or, if it exists, it might not be unique. However, there exist certain conditions (see for instance [CaK1, Proposition 8.3]) which imply the existence and uniqueness of a convolution. In [CKL1] we showed that the complex (12) defining $T_i \mathbf{1}_\lambda$ satisfies these conditions when $r = 0$ and hence has a unique convolution $C(T_i) \mathbf{1}_\lambda \in \mathcal{K}_{\text{Gr},m}$. Notice that $C(T_i)$ is now a 1-morphism in $\mathcal{K}_{\text{Gr},m}$ (not a complex of 1-morphisms). Thus T_ω^2 has a convolution $C(T_\omega^2) \in \mathcal{K}_{\text{Gr},m}$ and the map $\mathbf{1}_\lambda \rightarrow T_\omega^2 \mathbf{1}_\lambda$ induces a map $\mathbf{1}_\lambda \rightarrow C(T_\omega^2 \mathbf{1}_\lambda)$. Subsequently we can then consider the limit $\lim_{n \rightarrow \infty} C(T_\omega^2 \mathbf{1}_\lambda)^n$ and ask if it is well defined.

Lemma 11.5. *Suppose λ is a weight such that $T_i \mathbf{1}_\lambda = [E_i F_i \mathbf{1}_\lambda \langle -1 \rangle \rightarrow \mathbf{1}_\lambda]$ (*i.e.* $\langle \lambda, \alpha_i \rangle = 0$ and the weight spaces $\lambda + r\alpha_i$ are zero if $|r| > 1$). Then the limit $\lim_{n \rightarrow \infty} C(T_i)^{\pm 2n} \mathbf{1}_\lambda$ exists as a 1-morphism in $\mathcal{K}_{\text{Gr},m}^\pm$.*

Remark 11.6. Notice that although P^- (resp. P^+) is a complex unbounded *below* (resp. *above*), its convolution turns out to be a kernel unbounded *above* (resp. *below*).

Proof. The composition $T_i^{2n} \mathbf{1}_\lambda$ is homotopic to a complex of the form

$$(64) \quad [E_i F_i \mathbf{1}_\lambda \langle -2n-1 \rangle \rightarrow E_i F_i \mathbf{1}_\lambda \langle -2n+1 \rangle \rightarrow \cdots \rightarrow E_i F_i \mathbf{1}_\lambda \langle -1 \rangle \rightarrow \mathbf{1}_\lambda]$$

as described in equation (55). Thus $C(T_i^{2n} \mathbf{1}_\lambda)$ is equal to some convolution of this complex. It is not hard to check that in $\mathcal{K}_{\text{Gr},m}$ the kernel $E_i F_i \mathbf{1}_\lambda$ is actually a sheaf. Since $\langle 1 \rangle = [1]\{-1\}$ this means that

in $C(\mathbb{T}_i^{2n})\mathbf{1}_\lambda$, the term $\mathbf{E}_i\mathbf{F}_i\mathbf{1}_\lambda\langle -2k+1 \rangle$ in homological degree $-k$ contributes a sheaf in homological degree $-(-2k+1) - k = k-1$. Taking the limit we find that $\lim_{n \rightarrow \infty} C(\mathbb{T}_i)^{2n}\mathbf{1}_\lambda$ lies in $\mathcal{K}_{\text{Gr},m}^+$. \square

Note that in the proof of Lemma 11.5 we used that $\mathbf{E}_i\mathbf{F}_i\mathbf{1}_\lambda$ is a sheaf. In fact we only needed to know that $\mathbf{E}_i\mathbf{F}_i\mathbf{1}_\lambda$ is some bounded complex since then $\mathbf{E}_i\mathbf{F}_i\mathbf{1}_\lambda\langle -2k+1 \rangle$ contributes this bounded complex shifted up by k and as $k \rightarrow \infty$ this fixed complex drifts off to $+\infty$. Since $\mathbf{E}_i\mathbf{F}_i\mathbf{1}_\lambda$ is clearly bounded we only end up using that $\mathbb{T}_i^{2n}\mathbf{1}_\lambda$ is homotopic to the complex in (64) where the difference between the shift $\langle \cdot \rangle$ and the homological degree tends to $-\infty$. A more detailed study of $\mathbb{T}_\omega^{2n}\mathbf{1}_\lambda$ along the lines of section 3.4 should prove the following.

Conjecture 11.7. *The limit $\lim_{n \rightarrow \infty} C(\mathbb{T}_i)^{\pm 2n}\mathbf{1}_\lambda$ exists as a 1-morphism in $\mathcal{K}_{\text{Gr},m}^\pm$ and is isomorphic to a convolution of \mathbf{P}^\mp .*

11.3. Geometric construction of clasps. The clasp \mathbf{P}^- was defined algebraically in section 6 for any categorical 2-representation. However, the definition of the 2-representation $\mathcal{K}_{\text{Gr},m}$ from section 8 is almost entirely geometric using the affine Grassmannian.

Question. Does \mathbf{P}^- have a simpler description in terms of the geometry of the affine Grassmannian?

The answer seems to be “yes” when $m = 2$ and “perhaps not” for $m > 2$.

11.3.1. Case $m = 2$. Suppose $m = 2$ and $\underline{i} = (\underline{0}, 1, 1, \underline{2})$. Then $Y(\underline{i})$ is a compactification of $T^*\mathbb{P}^1$ which categorifies $V^{\otimes 2}$ (where V is the standard \mathfrak{sl}_2 -module). Now

$$Y(\underline{i}) \cong \{\mathbb{C}[z]^2 = L_0 \subset L_1 \subset L_2 \subset \mathbb{C}(z)^2 : zL_j \subset L_{j-1} \text{ and } \dim(L_j/L_{j-1}) = 1 \text{ for } j = 1, 2\}$$

so forgetting L_1 induces a natural map $p : Y(\underline{i}) \rightarrow \overline{Y(\underline{i})}$ where

$$\overline{Y(\underline{i})} = \{\mathbb{C}[z]^2 = L_0 \subset L_2 \subset \mathbb{C}(z)^2 : zL_2 \subset L_2, z^2L_2 \subset L_0 \text{ and } \dim(L_2/L_0) = 2\}.$$

Generically p is one-to-one since L_1 can be recovered as $L_1 = zL_2$, but over the point where $\{L_2 = z^{-1}L_0\} \in \overline{Y(\underline{i})}$ the fibre is \mathbb{P}^1 . Note that the restriction of p to the open subscheme $T^*\mathbb{P}^1 \subset Y(\underline{i})$ is just the affinization morphism. Subsequently, there is a natural functor

$$p^*p_* : D^-(Y(\underline{i})) \rightarrow D^-(\overline{Y(\underline{i})}).$$

We have to allow complexes unbounded below because $\overline{Y(\underline{i})}$ is singular and hence p^* can take a bounded complex to an unbounded below complex.

On the other hand, we have the 2-representation

$$D(Y(\underline{0}, 2, 0, \underline{2})) \begin{matrix} \xrightarrow{\mathbf{E}} \\ \xleftarrow{\mathbf{F}} \end{matrix} D(Y(\underline{i})) \begin{matrix} \xrightarrow{\mathbf{E}} \\ \xleftarrow{\mathbf{F}} \end{matrix} D(Y(\underline{0}, 0, 2, \underline{2}))$$

used to define the categorical clasp \mathbf{P}^- . To make the connection with geometry it is easier if we work with the dual clasp \mathbf{P}^+ . As discussed in section 11.2, \mathbf{P}^+ has a well defined convolution. This functor is induced by some kernel $C(\mathbf{P}^+) \in D^-(Y(\underline{i}) \times Y(\underline{i}))$. The following result was proved in section 6.3 of [Ca1] (Proposition 6.8).

Proposition 11.8. *If $m = 2$ and $\underline{i} = (\underline{0}, 1, 1, \underline{2})$ then the composition $p^*p_* : D^-(Y(\underline{i})) \rightarrow D^-(\overline{Y(\underline{i})})$ is induced by the kernel $C(\mathbf{P}^+) \in D^-(Y(\underline{i}) \times Y(\underline{i}))$.*

If you are interested in the geometry, then this result gives a representation theoretic way to understand the somewhat complicated functor p^*p_* in terms of the simpler functors \mathbf{E} and \mathbf{F} . For instance, it allows you to compute the cohomology of the kernel which induces p^*p_* . Unfortunately, as we explain next, it does not generalize in an obvious way.

11.3.2. *Case $m > 2$.* Suppose $m > 2$ and consider $\underline{i} = (\underline{0}, m-1, 1, \underline{m})$. This time $Y(\underline{i})$ compactifies $T^*\mathbb{P}^{m-1}$. Forgetting L_1 gives a projection $p : Y(\underline{i}) \rightarrow \overline{Y(\underline{i})}$ where $\overline{Y(\underline{i})}$ is the singular variety

$$\{\mathbb{C}[z]^m = L_0 \subset L_2 \subset \mathbb{C}(z)^m : zL_2 \subset L_2, z^2L_2 \subset L_0, \dim \ker(z|_{L_2/L_0}) \geq m-1, \dim(L_2/L_0) = m\}.$$

As above, restricting p to $T^*\mathbb{P}^{m-1} \subset Y(\underline{i})$ gives the affinization map.

On the other hand, we have the 2-representation

$$D(Y(\underline{0}, m, 0, \underline{m})) \xrightleftharpoons[\mathrm{F}]{\mathrm{E}} D(Y(\underline{i})) \xrightleftharpoons[\mathrm{F}]{\mathrm{E}} \dots \xrightleftharpoons[\mathrm{F}]{\mathrm{E}} D(Y(\underline{i}')) \xrightleftharpoons[\mathrm{F}]{\mathrm{E}} D(Y(\underline{0}, 0, m, \underline{m}))$$

where $\underline{i}' = (\underline{0}, 1, m-1, \underline{m})$. This gives rise to \mathbf{P}^+ which categorifies the composition

$$V_{\Lambda_1} \otimes V_{\Lambda_{m-1}} \xrightarrow{\pi} V_{\Lambda_1 + \Lambda_{m-1}} \xrightarrow{\iota} V_{\Lambda_1} \otimes V_{\Lambda_{m-1}}.$$

The complex \mathbf{P}^+ has a convolution $C(\mathbf{P}^+) \in D^-(Y(\underline{i}) \times Y(\underline{i}))$. Unfortunately, the analogue of Proposition 11.8 is no longer true. In other words:

Claim. If $m > 2$ and $\underline{i} = (\underline{0}, 1, m-1, \underline{m})$ then the composition $p^*p_* : D^-(Y(\underline{i})) \rightarrow D^-(Y(\underline{i}))$ is *not* induced by the kernel $C(\mathbf{P}^+)$.

To see this one checks that these two functors are not even equal at the level of K-theory. The kernel of $[\mathbf{P}^+] \in \mathrm{End}(V_{\Lambda_1} \otimes V_{\Lambda_{m-1}})$ is one-dimensional since $\dim(V_{\Lambda_1} \otimes V_{\Lambda_{m-1}}) - \dim(V_{\Lambda_1 + \Lambda_{m-1}}) = 1$. On the other hand, the kernel of $[p^*p_*]$ is at least $(m-1)$ -dimensional since p_* kills $\mathcal{O}_{\mathbb{P}^{m-1}}(-i)$ for $i = 1, 2, \dots, m-1$ where

$$\mathbb{P}^{m-1} = \{\mathbb{C}[z]^m = L_0 \subset L_1 \subset L_2 \subset \mathbb{C}(z)^m : L_2 = z^{-1}L_0\} \subset Y(\underline{i}).$$

Subsequently, there are two obvious questions if $m > 2$ and $\underline{i} = (\underline{0}, 1, m-1, \underline{m})$.

Question #1. Does the kernel $C(\mathbf{P}^+) \in D^-(Y(\underline{i}) \times Y(\underline{i}))$ have a geometric description?

Question #2. Is there a representation theoretic description of $p^*p_* : D^-(Y(\underline{i})) \rightarrow D^-(Y(\underline{i}))$?

However, there is still hope that all clasps can be explained geometrically when $m = 2$. More precisely, consider

$$D(Y(\underline{i})) \cong \{\mathbb{C}[z]^2 = L_0 \subset L_1 \subset \dots \subset L_k \subset \mathbb{C}(z)^2 : zL_{j+1} \subset L_j, \dim(L_{j+1}/L_j) = 1\}$$

where $\underline{i} = (\underline{0}, 1^k, \underline{2})$ for some $k \in \mathbb{N}$. Then, as above, we have a projection $p : Y(\underline{i}) \rightarrow \overline{Y(\underline{i})}$ which forgets L_1, \dots, L_{k-1} . On the other hand, we have the functor \mathbf{P}^+ which categorifies the clasp

$$V_{\Lambda_1}^{\otimes k} \xrightarrow{\pi} V_{k\Lambda_1} \xrightarrow{\iota} V_{\Lambda_1}^{\otimes k}$$

and its convolution $C(\mathbf{P}^+) \in D^-(Y(\underline{i}) \times Y(\underline{i}))$.

Conjecture 11.9. If $m = 2$ and $\underline{i} = (\underline{0}, 1^k, \underline{2})$ for some $k \in \mathbb{N}$ then the composition $p^*p_* : D^-(Y(\underline{i})) \rightarrow D^-(Y(\underline{i}))$ is induced by the kernel $C(\mathbf{P}^+) \in D^-(Y(\underline{i}) \times Y(\underline{i}))$.

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